A critique of the application of Fourier analysis to finite domain measurements and numerical simulations

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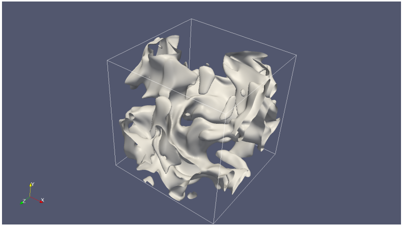
Motivation and Objectives

- Goals: to be able to generate a variety of turbulence spectra by changing the nature of the forcing.
- Methodology: alternative forcing scheme to the linear forcing of Lundgren (2003).
- Outcome: Performed DNS simulations to a 128^3 periodic box of length $0.128m$.
- Validation: tested for homogeneity and isotropy at various conditions.



Forcing Scheme to sustain HIT

Momentum

$$\frac{\partial \rho v_j}{\partial t} + \frac{\partial \rho v_j v_i}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial x_i} - \frac{\partial p}{\partial x_j} + S v_j + S_j^p$$

- v_i fluctuating velocity; S source term linear in the velocity; S_j^p additional source terms.
- Use part of a model spectrum to synthesize $v_{j,triggered}$ sampled from pseudo-random wavenumber vectors that satisfy continuity.

Forcing Scheme

$$S_j^p = \frac{\rho}{\Delta t} \frac{\sqrt{q^2_{wanted}} - \sqrt{q^2_{computed}}}{\sqrt{q^2_{wanted}}} v_{j,triggered}$$

TABLE: Tests for Homogeneity and Isotropy.

	Local	Global
Homogeneity	$\frac{\langle s_{ij}s_{ij}\rangle}{\langle \omega_i\omega_i\rangle} = \frac{1}{2}$	$B^1_{LL} = B^2_{LL} = B^3_{LL}$ and $B^1_{NN} = B^2_{NN} = B^3_{NN}$
Isotropy	$\langle[\frac{\partial u_1}{\partial x_1}]^2\rangle = \langle[\frac{\partial u_2}{\partial x_2}]^2\rangle = \langle[\frac{\partial u_3}{\partial x_3}]^2\rangle = \frac{1}{2}\langle[\frac{\partial u_1}{\partial x_2}]^2\rangle = \frac{1}{2}\langle[\frac{\partial u_2}{\partial x_1}]^2\rangle = \frac{1}{2}\langle[\frac{\partial u_1}{\partial x_3}]^2\rangle = \frac{1}{2}\langle[\frac{\partial u_3}{\partial x_1}]^2\rangle = \frac{1}{2}\langle[\frac{\partial u_2}{\partial x_3}]^2\rangle = \frac{1}{2}\langle[\frac{\partial u_3}{\partial x_2}]^2\rangle = -2\langle\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\rangle = -2\langle\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\rangle = -2\langle\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\rangle$	$B_{NN} = B_{LL} + \frac{1}{2}r\frac{dB_{LL}}{dr}$

- Local tests are not mentioned in the literature.

Results - Turbulence statistics

Table: Statistics at different triggering wavelengths.

Quantity	Low Range	Medium Range	High Range
	50.0 - 100.0	200.0 - 400.0	400.0 - 600.0
$\frac{1}{2}\langle u_1^2 \rangle$ $(m^2 s^{-2})$	0.00167	0.00217	0.00206
$\frac{1}{2}\langle u_2^2 \rangle$ $(m^2 s^{-2})$	0.00191	0.00216	0.00208
$\frac{1}{2}\langle u_3^2 \rangle$ $(m^2 s^{-2})$	0.00206	0.00215	0.00206
ϵ_f $(m^2 s^{-3})$	0.00566	0.03003	0.06622
L_{11}^1 (m)	0.03159	0.00932	0.00556
L_{22}^1 (m)	0.02042	0.00423	0.00274
L_{33}^1 (m)	0.01399	0.00439	0.00270
λ_{11}^1 (m)	0.01928	0.00806	0.00535
λ_{22}^1 (m)	0.01308	0.00571	0.00382
λ_{33}^1 (m)	0.01268	0.00569	0.00381
η_κ (m)	0.00087	0.00057	0.00047

RESULTS - 1D SPECTRA

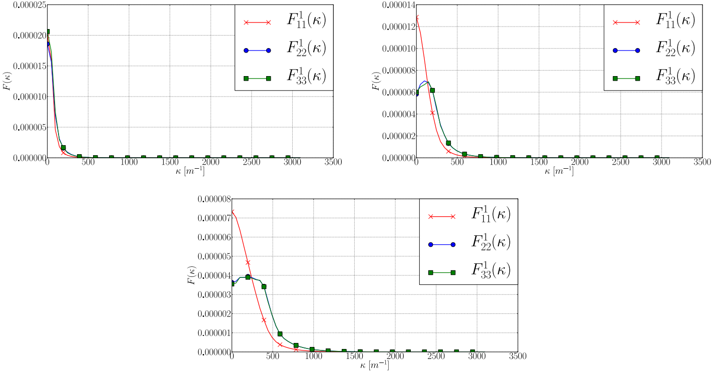


FIGURE: topleft: forcing 50 to 100; topright: 200 to 400; bottom: 400 to 600 (von Karman/Howarth spectrum near peak)

RESULTS - SPATIAL TWO POINT CORRELATIONS AND ISOTROPIC CONTINUITY

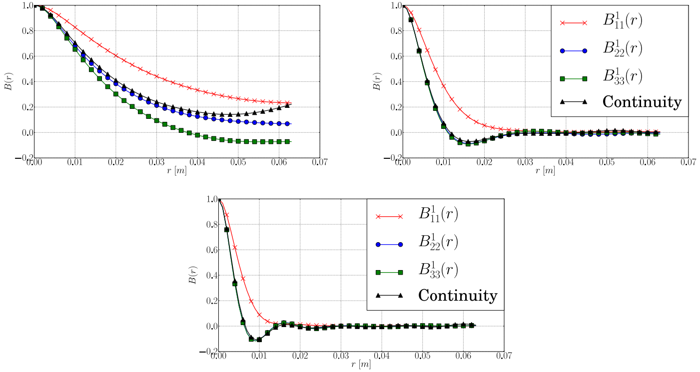
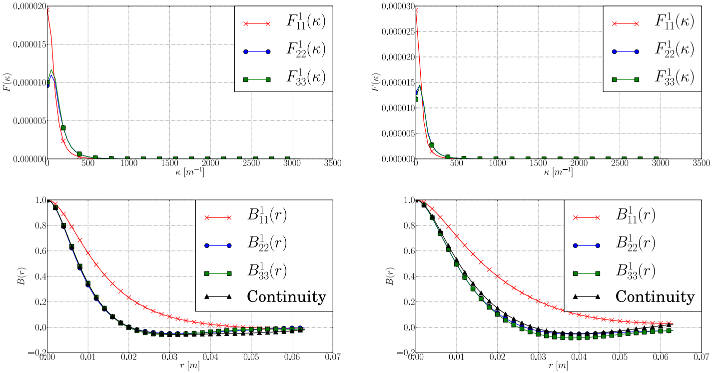


FIGURE: topleft: forcing 50 to 100; topright 200 to 400; bottom 400 to 600 (computed by assuming periodic)

Results for different input spectra - forcing at $100.0 \leq \kappa \leq 300.0$



How does one optimally (and correctly) decompose a field into orthogonal (hopefully) functions?

- Orthogonality allows us to talk about how individual eigenfunctions interact? E.G. Triads of wavenumbers, POD modes, etc. Not possible with 'regions of space' which all talk to each other (e.g. because of pressure, etc.).
- Answer was provided by Lumley in 1966. Seek maximal projection (in Rieman sense) of velocity (or anything else) on random field; i.e.,

$$\langle |\vec{u}(\vec{x},t) \cdot \vec{\phi}(\vec{x},t)| \rangle = \langle |\alpha|^2 \rangle = \lambda \tag{1}$$

How does one optimally (and correctly) decompose a field into orthogonal (hopefully) functions? ... continued

- Result is following integral equation (if it exists):

$$\int_{allspaceandtime} \langle u_i(\vec{x},t) u_j(\vec{x}',t')\rangle \phi_j(\vec{x}',t) d\vec{x}' dt' = \lambda \phi_i(\vec{x},t) \quad (2)$$

- The $\phi_i(\vec{x},t)$ are deterministic and the best one can do (at least in terms of the energy).
- There are several KNOWN GENERAL SOLUTIONS
  1. If flow is truly homogenous or stationary (infinite extent and statistics independent of origin), then the ϕ_i's are continuous Fourier modes of wavenumber $\vec{k}$ (and frequency ω), and λ is the three (or four) dimensional spectrum, say $F_{i,j}(\vec{k},\omega)$. I.e. **Fourier Transforms**
  2. If the flow is periodic (or periodic homogeneous), then the ϕ's are also Fourier modes, but only at integer multiples of 2π/period. I.e., **Fourier Series.**

Fourier Analysis: What is did we really compute?

- In a one-dimensional homogeneous flow the Fourier transform *in the sense of generalized functions*, say $\hat{u}(k)$, is defined as (c.f. Lumley 1970 *Stochastic Tools in Turbulence*, George (2011) *Lectures in Turbulence for the 21st Century* [www.turbulence-online.com]):

$$\hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}u(x)dx \qquad (3)$$

where $u(x)$ is our random variable (e.g., velocity) and:

$$\langle \hat{u}^*(k')\hat{u}(k)\rangle dkdk' = F(k)\delta(k'-k)dkdk' \qquad (4)$$

- $F(k)$ is the spectrum defined by:

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikr}B(r)dr \qquad (5)$$

where $B(r) = \langle u(x)u(x+r)\rangle$ is the two-point correlation.

- In the real world we can only see a finite piece of this, say $(-L/2, L/2)$ or $(0, L)$. So the best we can do is:

$$\hat{u}_L(k) = \frac{1}{2\pi}\int_{-L/2}^{L/2} e^{-ikx}u(x)dx \tag{6}$$

- We create a spectral estimator which converges to the right answer in the limit as $L \to \infty$:

$$F_L(k) = \frac{2\pi}{L}\langle|\hat{u}_L(k)|^2\rangle \tag{7}$$

which actually gives us this:

$$F_L(k) = \frac{1}{2\pi}\int_{-L}^{L} e^{-ikr}B(r)[1 - |r|/L]dr \tag{8}$$

- But for *finite* L it does not. Our estimator is 'windowed' and information is 'leaked' among frequencies. This usually shows up at lower spectral values and is often confused with 'noise'.

EXAMPLE SPECTRA USING LOW-PASS FILTERED NOISE

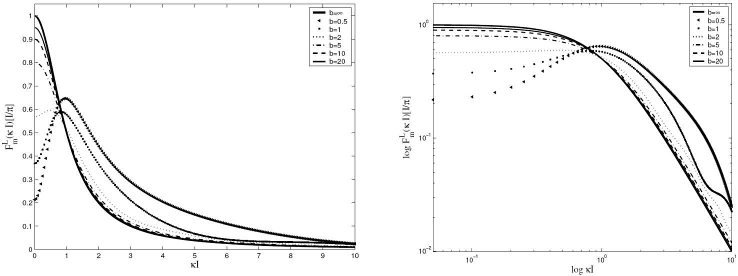


FIGURE: Exact solutions. $b = L/l$, ratio of domain to integral scale. Need about 20 integral scales to make reasonable spectrum. (from Wänström PhD dissertation 2009).

Same thinking applies if vector field

1D Spectra Estimators

$$F_{ijL}(\kappa) = \frac{2\pi}{L}\langle \hat{u}_{iL}(\kappa)\hat{u}^{\star}_{jL}(\kappa)\rangle$$

- where $F_{ijL}(\kappa)$ is the 1D spectrum and * is the complex conjugate.
- subscript L states that Fourier Transform is over a finite domain.
- Hence, the spectra are estimators because they are convolved (*contaminated*) with a window.
- This is important when trying to obtain two-point correlations.

Two point correlations: need to remove window if inverse transform

Two-point correlations from finite estimator

$$B_{ij}(r)\left(1-\frac{|r|}{L}\right)=\frac{1}{2\pi}\int_{-L}^{L}e^{+i2\pi\kappa r}F_{ijL}(\kappa)\,\mathrm{d}\kappa$$

- Inverse Fourier Transform $F_{ijL}(\kappa)$.
- $B_{ij}(r)$ is the two-point correlation.
- $\left(1-\frac{|r|}{L}\right)$ is the window funtion.
- We must divide by the window function to get $B_{ij}(r)$.
- BUT **what if our flow is NOT really homogeneous?**

What if flow is periodic, say with period L?

- Now we need to use **Fourier Series.**

$$u(x) = \sum_{m=-\infty}^{\infty} C_m e^{+i2\pi mx/L} \tag{9}$$

where the **Fourier Series coefficients** are defined by:

$$C_m = \frac{2\pi}{L} \int_0^L e^{-i2\pi mx/L} u(x) dx \tag{10}$$

- The correlation and energy are given by:

$$B(r) \quad = \quad \sum_{m=-\infty}^{\infty} e^{+i2\pi mr/L} |C_m|^2 \tag{11}$$

- The $|C_m|^2$ form a Fourier **LINE** spectrum. Sometimes people multiply by $L/2\pi$ and *PRETEND* that this is a spectral density. It is not!

Why has this caused so much confusion?

- Answer lies in the way we do numerical analysis, both in simulations and in data analysis.
- For a continuous homogenous process we approximate $\hat{u}(k)$ by only N realizations, $u(n\Delta x)$, with spacing $\Delta x = L/N$; i.e.,

$$
\begin{aligned}
\hat{u}_L(k) &= \frac{1}{2\pi}\int_0^L e^{-ikx}u(x)dx \\
&\approx \frac{1}{2\pi}\sum_0^{N-1} e^{ikn\Delta x}u(n\Delta x)\Delta x \\
&= \frac{1}{2\pi}\frac{L}{N}\sum_0^{N-1} e^{iknL/N}u_n
\end{aligned}
$$

Why has this caused so much confusion? ... continued

- For convenience we choose $k = 2\pi m/L$ to obtain N independent Fourier **Transform** coefficients as:

$$\hat{u}(m/L) = \hat{u}_m = \frac{L}{2\pi}\left[\frac{1}{N}\sum_{0}^{N-1} e^{i2\pi mn/N} u_n\right] \qquad (15)$$

- But the term in square bracket is **EXACTLY** the **Finite Fourier Series** result *IF* we *imagined* the same record to be periodically repeated with period L.
- BUT it is NOT a periodic signal – at least not if if is just a piece of a homogeneous (or nearly homogeneous) field.
- UNLESS it is in fact a periodic field (and therefore by definition NOT homogenous).

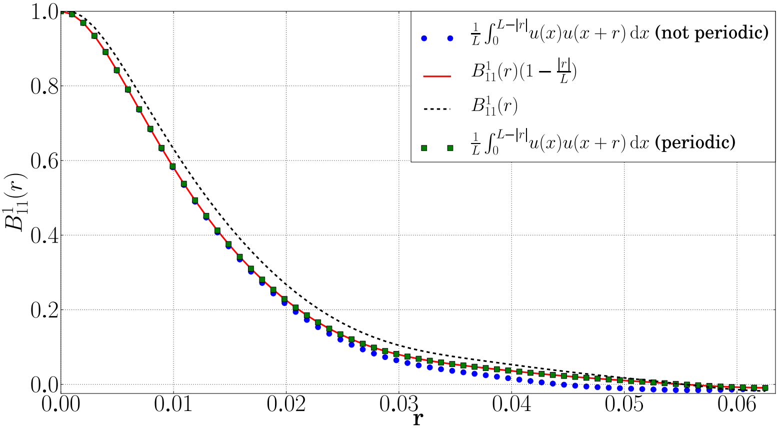


FIGURE: Window removal gives wrong answer. Flow knows it is periodic.

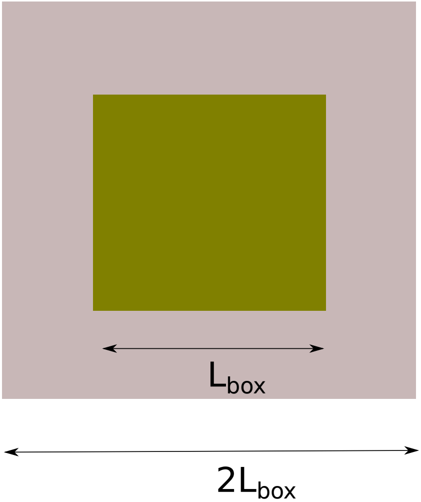


FIGURE: Can we destroy periodicity and make flow behave as homogenous?

Answer appears to be yes. Correlations and spectra on inner box.

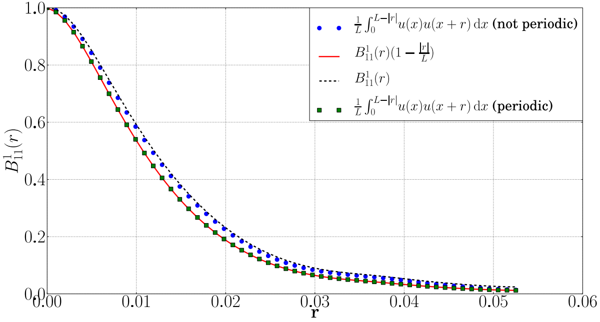


FIGURE: This is the reverse of the previous result. Window removal gives correct answer.

Conclusions

- Proposed a forcing scheme alternative to linear forcing.
- Used various triggering wavenumbers and tested for local and global homogeneity and isotropy.
- All test-cases were locally homogeneous and isotropic
- Periodicity *kills* the window-effect. If periodicity is removed artificially the window-effect appears and must be removed. Bottom line: The simulated flow knew it was NOT homogeneous, but periodic.
- Time signals from forced turbulence are the exact opposite – stationary, NOT periodic. Need window *if handled correctly*.
- Which is right answer? Can flow be used to test theories of homogeneous turbulence? Not clear in general – so far at least.