Turbulence Homework 3

Stochastic Tools in Turbulence, AEM-ADV18
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You will find almost all the answers in Appendices C, D and E of WKG's 'Lectures in Turbulence for the 21st Century. Don't just agree with them. Learn to do them yourselves!

1. What is a Fourier Series representation? What kind of signals can it be applied to? How do you find the Fourier Series coefficients?

2. Consider a saw-tooth wave defined in the following manner.

$$u(t) = t - nT_p \tag{1}$$

where T_p is the period and n is an interger for which $-\infty < n < \infty$. Find its Fourier Series representation and plot how the signal is built up as each new Fourier mode is added.

3. Do the same steps for a square wave defined by :

$$\begin{aligned} u(t) &= 1 \qquad\qquad ; 0 \leq t \leq T_p/2 \\ &= 0 \qquad\qquad ; T_p/2 \leq t \leq T_p \end{aligned} \tag{2}$$

4. What is a Fourier transform? What is the inverse Fourier Transform?

5. Compute the Fourier Transform of the so-called top-hat function given by:

$$\begin{aligned} u(t) &= 1 \qquad\qquad -T/2 \leq t \leq T \\ &= 0 \qquad\qquad |t| \geq T \end{aligned} \tag{3}$$

This is a very important FT, since it appears often as a window function. Try expressing your answer as a sinc function

6. What is the Fourier Transform of the triangle function (or Barlett window) defined by:

$$\begin{aligned} u(t) &= 1 - |t|/T \qquad\qquad |t| \leq T \\ &= 0 \qquad\qquad |t| > T \end{aligned} \tag{4}$$

This too is a very important window function, so remember it. See if you can express you answer in terms of the sinc function.

7. Compute the inverse Fourier transform coefficients of:

$$\hat{u}(f) = \sqrt{2\pi T} exp(-(2\pi f T)^2/2). \tag{5}$$

(Hint: you need to complete the square in the integration. And it helps if you remember that the integral of $(1/\sqrt{2\pi T}) exp(-t^2/2T^2)$ is just unity.) These are an important Fourier transform pair, so remember them.

8. What is the problem taking the Fourier Transform *in the ordinary sense* of $u(t) = 1$? Show how you can define a generalized function that recovers $u(t)$ in the limit. And then show how it can be used to compute the Fourier Transform of 1 *in the sense of generalized functions*. What is our shortcut notation for all of this?

9. Now use you knowledge of generalized functions to derive in the same way as above the Fourier Transform of $exp(+i2\pi f_o t)$ where f_o is a fixed frequency. And show how you can represent all of this using the same functional symbol as above. What would be the Fourier Transform of $u(t) = exp(-i2\pi f_o t)$?

10. Use all the things you have learned above to compute the Fourier Transform of the string of delta functions given by:

$$g_s(t) = \sum_{n=-\infty}^{\infty} \Delta t \; \delta(t - n/\Delta t) \tag{6}$$

where Δt is the time between pulses.