Similarity Analysis of the Two-Point Velocity Correlation Tensor in a Turbulent Axisymmetric Jet

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ABSTRACT

Previous analysis of the far field in a turbulent, incompressible, isothermal jet has demonstrated that the Reynolds averaged equations which govern the evolution of the single-point statistical moments admit to similarity solutions. This analysis demonstrates that the equations which govern the evolution of the two-point velocity correlation tensor admit to similarity solutions. The final similarity equations are not independent of the growth rate or Reynolds number of the jet, indicating that the form of the similarity solution may be dependent on the source conditions.

Keywords *similarity, two-point velocity correlation*

INTRODUCTION

There has been a long history of using similarity solutions in the study of laminar fluid dynamics (Batchelor, 1967) dating back to the early work on the laminar boundary layer by Blasius. Physically, the similarity hypothesis is useful because it sheds insight into the nature of the evolution of the flow. Mathematically, the technique is also useful for analyzing problems because it reduces the number of independent variables in the problem by one or more, and can often be utilized to reduce a problem governed by a partial differential equation to one which is governed by an ordinary one.

Similarity analyses have also been utilized to an examine an extensive number of turbulent flows. A review of some of these earlier analyses can be found in classical texts on turbulence such as Tennekes and Lumley (1972) or Hinze (1975). Traditionally, the similarity hypothesis was applied to a turbulent problem by choosing scales for the statistical moments in the governing equations using a single length and velocity scale. The similarity solutions were then substituted into these equations to determine if the hypothesized similarity solutions were consistent with the equations. George (1989) pointed out that this approach may over-constrain the analysis and suggested using arbitrary scales for the statistical moments in the equations. The equations of motion are then used to determine what relationships must exist between the scales for the moments in order to ensure that the equations of motion admit to a similarity solution. Since the method suggested by George (1989, 1994) is a more general approach which does not exclude solutions of the traditional type, it is utilized in this analysis to query if the equations which govern the two-point velocity correlation tensor admit to similarity solutions.

Almost all previous attempts to find similarity solutions in turbulent flows have centered on the Reynolds averaged equations which govern the single point statistical moments (i.e., moments in which all of the variables are evaluated at the same point in space and time). The notable exceptions are the investigations of decaying homogeneous isotropic turbulence (v. von Karman and Howarth 1941, Batchelor 1948,   and George 1992) and the investigation of the homogeneous shear layer reported by George and Gibson (1992). The authors of this investigation are not aware of any other attempts to find similarity solutions of the for the equations which govern two-point velocity correlation tensor in a non-homogeneous shear flow.

This analysis investigates whether similarity solutions exist for the equations which govern the two-point velocity correlation tensor in a single spatially-evolving flow: the far field of the axisymmetric jet. At present there isn’t sufficient experimental evidence to test the similarity hypothesis for the two-point velocity correlation tensor, so no attempt is made to pursue this important question in this analysis. This issue will be addressed in a future investigation. The extension of the theory to other spatially-evolving flows is discussed elsewhere (Ewing, 1995).

REVIEW OF SINGLE-POINT SIMILARITY HYPOTHESIS

It is useful to briefly review the application of the similarity analysis to the equations which govern the evolution of the single-point statistical moments in the far field of an axisymmetric jet; in order to dem
onstrate the technique. The similarity analysis of the single-point equations for the far field of the axisymmetric jet was reported previously by George (1989), but recent studies of other shear flows (v. George, 1994) have illustrated the need to vary the earlier approach outlined by George(1989). The derivation in this section is in keeping with this later work. The similarity analysis is carried out using only the highest order terms in the Reynolds average equations, so the solutions outlined in this section are first order accurate.

Using standard thin-shear-layer and high-Reynolds-number assumptions (v. Tennekes and Lumley, 1972), the first order differential and integral equations for the mean momentum are given by (Hussein *et al.*, 1995)

$$U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} = -\frac{1}{x_2} \frac{\partial x_2 \overline{u_1 u_2}}{\partial x_2} \tag{1}$$

and

$$\int_0^\infty U_1^2 \, x_2 dx_2 = M_o \tag{2}$$

where U_i and u_i are the mean and fluctuating velocity respectively in the x_i direction. The geometry of the axisymmetric jet and orientation of the coordinate system are illustrated in figure 1. The over bar indicates an ensemble average and M_0 is the rate of momentum addition at the source.

It is hypothesized that a similarity solution exists to these equations in which the mean streamwise velocity, U_1, can be written as

$$U_1(x_1, x_2) = U_s(x_1) f(\eta), \quad \eta = \frac{x_2}{\delta(x_1)} \tag{3}$$

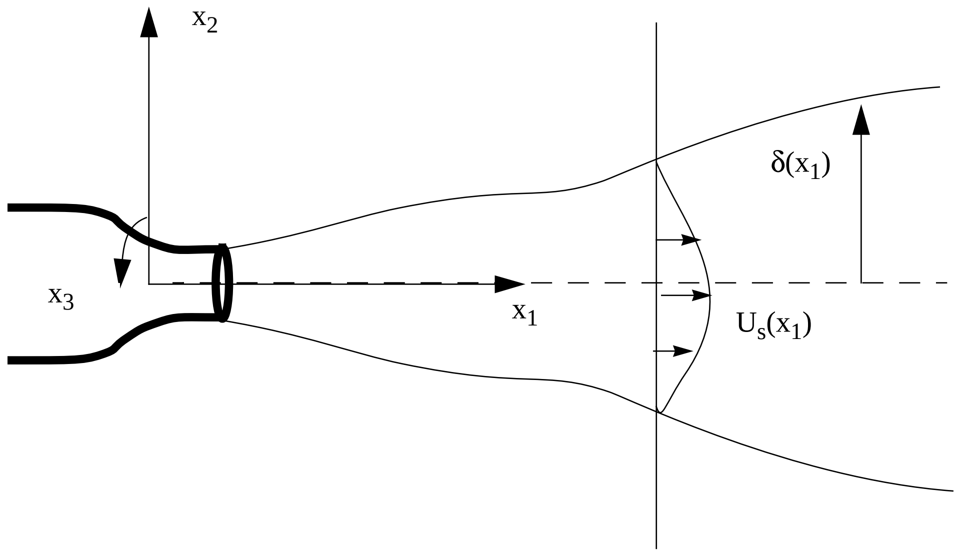

Figure 1. Geometry of the Far Field in the Axisymmetric Jet.

where U_s is a velocity scale and δ is a length scale for the far field of the axisymmetric jet.

The mean cross-stream velocity, U_2, can be determined by integrating the averaged incompressible continuity equation for this flow; i.e.

$$\frac{\partial U_1}{\partial x_1} + \frac{1}{x_2}\frac{\partial x_2 U_2}{\partial x_2} = 0 \tag{4}$$

Substituting the hypothesized similarity solution for U_1 (i.e. equ. 3) into equation 4 and integrating the equation from the center line to η yields

$$U_2(x_1, x_2) = -\left\{\left[\delta\frac{dU_s}{dx}\right] + 2\left[U_s\frac{d\delta}{dx}\right]\right\}\frac{1}{\eta}\int_0^{\eta}\tilde{\eta} f(\tilde{\eta})\, d\tilde{\eta} + \left[U_s\frac{d\delta}{dx}\right]\eta f(\eta) \tag{5}$$

Following George (1989), the hypothesized similarity solution for the Reynolds stress $\overline{u_1 u_2}$ is given by

$$\overline{u_1 u_2} = R_s(x_1)\, g(\eta) \tag{6}$$

In order to avoid over constraining the analysis of the problem the scale for the $\overline{u_1 u_2}$ is arbitrary at this point and is not chosen equal to U_s^2.

Substituting the hypothesized form of the similarity solutions (i.e., equation 3, 5, and 6) into equations 1 and 2 yields

$$\left[U_s\frac{dU_s}{dx_1}\right]f^2 - \left\{\left[U_s\frac{dU_s}{dx_1}\right] + 2\left[\frac{U_s^2}{\delta}\frac{d\delta}{dx_1}\right]\right\}\frac{1}{\eta}\frac{df}{d\eta}\int_0^{\eta}\tilde{\eta} f(\tilde{\eta})\, d\tilde{\eta} = -\left[\frac{R_s}{\delta}\right]\frac{1}{\eta}\frac{d\eta g}{d\eta} \tag{7}$$

and

$$[(U_s\delta)^2]\int_0^\infty f^2\tilde{\eta}\,d\tilde{\eta} = M_o \tag{8}$$

The x_1 dependence of each terms in equations 7 and 8 is contained in square brackets.

Mathematically, the evolution of the single-point moments governed by equations 7 and 8 are consistent with a similarity hypothesis if the x_1 dependence of all of the terms in each equation are proportional, so the x_1 dependence can be removed from equations 7 and 8. Physically, applying a similarity hypothesis to the single-point equations, implies that the flow evolves such that all of the terms in each equation maintain a relative balance as the flow evolves downstream; i.e., no term increases or decreases in size relative to the rest of the terms in the equation.

Thus, equations 7 and 8 admit to similarity solutions if

$$R_s \sim U_s^2\frac{d\delta}{dx_1}, \quad U_s \sim \frac{\sqrt{M_o}}{\delta}, \quad \frac{1}{\delta}\frac{d\delta}{dx_1} \sim \frac{1}{U_s}\frac{dU_s}{dx_1} \tag{9}$$

Note, the growth rate of the layer is not determined by these constraints because the ratio R_s/U_s^2 is not yet known. The second constraint can be only satisfied if the Reynolds number defined using the similarity velocity and length scales is a constant; i.e.,

$$Re = \frac{U_s\delta}{\nu} \sim \frac{\sqrt{M_o}}{\nu} \tag{10}$$

In this case, the third constraint in equations 9 is also satisfied.

In order to determine the growth rate of the jet, George (1989) examined the turbulent kinetic energy equation using scales for the rate of dissipation of the turbulent kinetic energy determined from physical arguments. Later analysis (*e.g.* George, 1994) indicated that it is more appropriate to examine the individual Reynolds stress component equations. The analyses of these equations for the far field of the axisymmetric jet are briefly outlined.

The first order equation for $\overline{u_1u_1}$ is given by

$$U_1\frac{\partial\overline{u_1^2}}{\partial x_1} + U_2\frac{\partial\overline{u_1^2}}{\partial x_2} = 2\overline{\frac{p}{\rho}\frac{\partial u_1}{\partial x_1}} - \frac{1}{x_2}\frac{\partial x_2\overline{u_1^2u_2}}{\partial x_2} - 2\overline{u_1u_2}\frac{\partial U_1}{\partial x_2} - \varepsilon_{u_1} \tag{11}$$

where ε_{u_1} is the dissipation rate of $\overline{u_1u_1}$. By utilizing similarity solutions for the new moments such as

$$\frac{\overline{u_1^2}}{2} = K_{u_1}(x_1)\,k_{u_1}(\eta), \quad \overline{\frac{p}{\rho}\frac{\partial u_1}{dx_1}} = P_{u_1}(x_1)\,p_{u_1}(\eta)$$

$$\overline{u_1^2u_2} = T_{u_1^2u_2}(x_1)\,t_{u_1^2u_2}(\eta), \quad \varepsilon_{u_1} = D_{u_1}(x_1)\,d_{u_1}(\eta) \tag{12}$$

It is straight forward to demonstrate that a similarity solution can exist for equation 11 only if

$$K_{u_1} \sim U_s^2,\ \ D_{u_1} \sim P_{u_1} \sim \frac{U_s^3}{\delta}\frac{d\delta}{dx_1},\quad T_{u_1^2 u_2} \sim U_s^3 \frac{d\delta}{dx_1} \tag{13}$$

Similar analyses can also be carried out for first order which govern the evolution of the moments $\overline{u_2 u_2}$, $\overline{u_3 u_3}$, and $\overline{u_1 u_2}$;i.e,

$$U_k \frac{\partial \overline{u_2^2}}{\partial x_k} = 2\overline{\frac{p}{\rho}\frac{1}{x_2}\frac{\partial x_2 u_2}{dx_2}} - \frac{1}{x_2}\frac{\partial}{dx_2}\left[x_2\overline{u_2^3} + 2x_2\frac{\overline{pu_2}}{\rho}\right] + \frac{2\overline{u_3^2 u_2}}{x_2} - \varepsilon_{u_2}, \tag{14}$$

$$U_k \frac{\partial \overline{u_3^2}}{\partial x_k} + 2\frac{U_2 \overline{u_3 u_3}}{x_2} = 2\overline{\frac{p}{\rho}\frac{1}{x_2}\frac{\partial u_3}{dx_3}} - \frac{1}{x_2}\frac{\partial x_2 \overline{u_2 u_3^2}}{\partial x_2} - \frac{2\overline{u_2 u_3^2}}{x_2} - \varepsilon_{u_3}, \tag{15}$$

and

$$U_k \frac{\partial \overline{u_1 u_2}}{\partial x_k} = \overline{\frac{p}{\rho}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)} - \frac{1}{x_2}\frac{\partial x_2 \overline{u_2^2 u_1}}{\partial x_2} + \frac{\overline{u_3^2 u_1}}{x_2} - \overline{u_2^2}\frac{\partial U_1}{\partial x_2}. \tag{16}$$

Expressing each of the moments in these equations in a similarity form analogous to those in equation 12, it is straightforward to demonstrate that these equations only admit to similarity solutions when

$$P_{u_2} \sim D_{u_2} \sim \frac{U_s K_v}{\delta}\frac{d\delta}{dx_1},\qquad T_{u_2^3} \sim T_{u_3^2 u_2} \sim U_s K_{u_2}\frac{d\delta}{dx_1} \tag{17}$$

$$P_{u_3} \sim D_{u_3} \sim \frac{U_s K_{u_3}}{\delta}\frac{d\delta}{dx_1},\qquad T_{u_2^2 u_3} \sim U_s K_{u_3}\frac{d\delta}{dx_1} \tag{18}$$

$$P_{u_1 u_2} \sim \frac{R_s U_s}{\delta}\frac{d\delta}{dx_1},\quad K_{u_2} \sim R_s \frac{d\delta}{dx_1},\quad T_{u_2^2 u_1} \sim T_{u_3^2 u_1} \sim R_s U_s \frac{d\delta}{dx_1} \tag{19}$$

In addition, the sum of the pressure-strain terms in the equations for the $\overline{u_1 u_1}$, $\overline{u_2 u_2}$, and $\overline{u_3 u_3}$ Reynolds stresses must be zero when the flow is incompressible; i.e.,

$$\overline{\frac{p}{\rho}\left(\frac{\partial u_1}{\partial x_1} + \frac{1}{x_2}\frac{\partial x_2 u_2}{\partial x_2} + \frac{1}{x_2}\frac{\partial u_3}{\partial x_3}\right)} = 0 \tag{20}$$

It is straightforward to demonstrate that this equation is only consistent with the hypothesize similarity solutions when

$$P_{u_1} \sim P_{u_2} \sim P_{u_3} \tag{21}$$

Thus using the conditions in equations 13, 17, and 19 it follows that

$$P_{u_1} \sim P_{u_2} \rightarrow K_{u_1} \sim K_{u_2} \sim U_s^2 \qquad K_{u_2} \sim R_s \frac{d\delta}{dx_1} \rightarrow K_{u_2} \sim U_s^2\left(\frac{d\delta}{dx_1}\right)^2 \tag{22}$$

Therefore, the equation of motion admit to a similarity solution only when

$$U_s^2 \sim U_s^2\left(\frac{d\delta}{dx_1}\right)^2 \rightarrow \frac{d\delta}{dx_1} \sim const \tag{23}$$

which is the result George (1994) determined for the plane jet.

Note, there are two different choices for the scale K_{u_2} in equation 22, U_s^2 and $U_s^2\,(d\delta/dx_1)^2$. It is not immediately clear from this analysis which of these choices is more appropriate when scaling data from different jets. One consequence of having these two choices is the growth rate of the jet appears as a parameter in the final set of equations which govern the similarity solutions. Therefore, the functional form of the similarity solutions (e.g. $k_{u_2}(\eta)$) may implicitly depend on the growth rate of the jet, or more properly on the initial conditions of the jet (v. George, 1989).

The constraints in equations 13, 17, an d 18 also define the scale for all of the dissipation rate terms in the Reynolds stress equations given by

$$D_s \sim \frac{U_s^3}{\delta} \tag{24}$$

The constant of proportionality in this equation may differ for the individual components and may depend on the source condition of the jet. George (1994) argued this conditions is satisfied for flows with either a large turbulent Reynolds number or flows with a constant turbulent Reynolds number. The analysis of equations which govern the evolution of the two-point velocity correlation tensor will shed further insight into why this second scaling argument is valid.

This analysis has demonstrated that the similarity solutions are consistent with the first order equations of motion. Experimental evidence must be examined to determine if actual flows evolve in this manner. The experimental data from the far field of the jet indicate that these flows do evolve in a manner which is consistent with the single-point similarity hypothesis (Hussein *et al.* 1994 or Panchapakesan and Lumley 1993).

SIMILARITY ANALYSIS OF THE TWO-POINT EQUATIONS

The single-point similarity analysis is useful because it supplies information about the evolution of the turbulent flow. However, more insight into the structure of the turbulent flow can be gained by examining the information contained in the two-point velocity correlation tensor. Assuming the turbulence to be statistically stationary, the equations which governs the evolution of the velocity correlation at two arbitrary points in a cylindrical coordinate system and a single point in time are given by (the analogous equation in cartesian coordinates is reported in Hinze, 1975)

$$
\begin{aligned}
&U_k \frac{\partial \overline{u_i u'_j}}{h^k \partial x_k} + U_k' \frac{\partial \overline{u_i u'_j}}{h^{k'} \partial x'_k} + \frac{U_2 \overline{u_3 u_j'}}{x_2} \delta_{i3} + \frac{U'_2 \overline{u'_3 u_i}}{x'_2} \delta_{j3} = -\frac{\partial \overline{u_k u_i u_j'}}{h^k \partial x_k} - \frac{\partial \overline{u'_k u_i u_j'}}{h^{k'} \partial x'_k} - \frac{\overline{u_2 u_i u_j'}}{x_2} - \frac{\overline{u'_2 u_i u_j'}}{x'_2} \\
&+ \frac{\overline{u_3^2 u'_j}}{x_2} \delta_{i2} + \frac{\overline{u'^2_3 u_i}}{x'_2} \delta_{j2} - \frac{\overline{u_2 u_3 u'_j}}{x_2} \delta_{i3} - \frac{\overline{u'_2 u'_3 u_i}}{x_2} \delta_{j3} - - \frac{1}{\rho h^{j'}} \frac{\partial \overline{p' u_i}}{\partial x_j'} - \frac{1}{\rho h^i} \frac{\partial \overline{p u_j'}}{\partial x_i} - \overline{u'_j u_k} \frac{\partial U_i}{\partial x_k} - \overline{u_i u'_k} \frac{\partial U'_j}{\partial x'_k} \\
&+ \nu \nabla^2 \overline{u_i u_j'} + \nu \nabla'^2 \overline{u_i u_j'} - \nu \left(\frac{\overline{u_2 u_j'}}{x_2^2} + \frac{2}{x_2^2} \frac{\partial \overline{u_3 u_j'}}{\partial x_3} \right) \delta_{i2} + \nu \left(\frac{2}{x_2^2} \frac{\partial \overline{u_2 u'_j}}{\partial x_3} - \frac{\overline{u_3 u_j'}}{x_2^2} \right) \delta_{i3} \\
&- \nu \left(\frac{\overline{u_2' u_i}}{x'^2_2} + \frac{2}{x'^2_2} \frac{\partial \overline{u_3' u_i}}{\partial x'_3} \right) \delta_{j2} + \nu \left(\frac{2}{x'^2_2} \frac{\partial \overline{u_2' u_i}}{\partial x'_3} - \frac{\overline{u_3' u_i}}{x'^2_2} \right) \delta_{j3}
\end{aligned} \tag{25}
$$

The prime in equation 25 is used to identify variables which are evaluated at one point in space, while the unprimed variables are evaluated at the second point (v. figure 2), while $h^j = (1,1,x_2)$ is the metric of the coordinate system. The value of the superscript on the metric, j, has the same value as the index of the differential coordinate next to the metric.

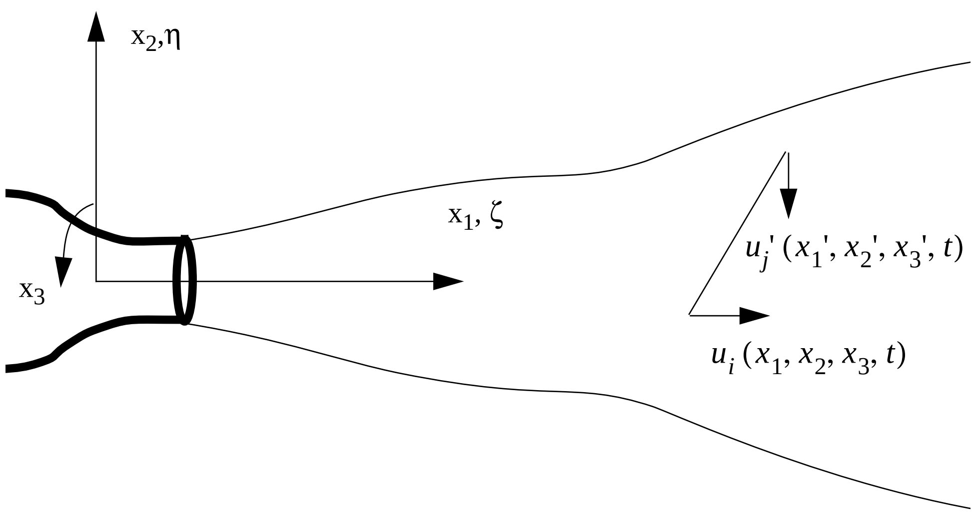

Figure 2. Two-Point Velocity Correlation in the Jet

The objective of this analysis is to determine if the set of equations which govern the evolution of the two-point velocity correlation tensor, equation 25, admit to a similarity solution for arbitrary separation between the two points. That is, do the equations of motion allow for solutions where the flow evolves such that the correlation of the velocities at the points x_k and x_k', $\overline{u_i u_j'}$, can be related to the correlation of the velocities at two other point downstream at x_k'' and x_k'''. In order to answer this question, the two-point correlations in equation 25 are written in a form where the functional dependence is the product of two parts (analogous to the similarity solutions for the single-point equations). The first part is a scale function which depends on the position of the two points relative to the origin of the jet, while the second part is a function of the separation vector between the points and independent of the position of the two points relative to the origin of the jet. This hypothesized set of solutions is then substituted into equation 25 to determine if it is consistent with this equation.

The first part of the similarity solution is a scale which indicates how the two-point velocity correlation varies as a function of the location of the downstream location of the two points. Analogous to the single-point analysis, the scales for the two-point correlations are determined by examining the restrictions imposed by the equations of motion. The scales for the two point correlation tensor must also agree with the scale determined from the single-point analysis in the limit of zero separation distance.

When considering the second portion of the similarity solution, which is a function of the separation distance between the points, it is important to note that the characteristic length scales of the turbulent field in the jet grow as the jet evolves downstream. In the single-point analysis this is accounted for by scaling the distance in radial direction by the length scale δ. The single-point similarity hypothesis also indicates that the Reynolds number, $Re=U_s\delta/\nu$, for the axisymmetric jet is a constant. Therefore, all of the physical length scales of the flow, such as the integral length scales and the Kolmogorov length scale, should grow in proportion as the flow evolves downstream. Consequently, it should be possible to normalize all of the length scales by one length scale. Of course, the solution for the statistical moments in the jet may depend on the ratio of the other length scales and the single length scale chosen to normalize the problem, but this ratio remains constant for any particular jet.

In order to rescale the separation distances and remove the influence of the length scale growth, a new coordinate system is defined by normalizing the differential lengths of the physical co-ordinate system by the local length scale $\delta(x_1)$; i.e.,

$$d\zeta \sim \frac{dx_1}{\delta(x_1)}, \quad d\eta \sim \frac{dx_2}{\delta(x_1)}, \quad \eta dx_3 \sim \frac{x_2 dx_3}{\delta(x_1)} \tag{26}$$

Clearly, the transformed coordinate system in the x_2 direction is η, the same coordinate which was used in single-point analysis. In addition, since the variation of the length in the azimuthal direction is contained in the metric x_2, the coordinate x_3 is the same in both the physical and transformed coordinate system. When the growth of the length scale δ is linear, the transformed coordinate in the mean flow direction is given by

$$\zeta = \ln\{(x_1 - x_1^v)/l\} \tag{27}$$

where l is a constant with units of length included for dimensional reasons and x_1^v is the location of the virtual origin of the jet. This transformation converts the streamwise coordinate in the jet from a semi-infinite coordinate x_1 to a coordinate ζ which is infinite in extent.

For functions which are dependent on two different positions in this new co-ordinate, say ζ and ζ', it is useful to define two new co-ordinates given by

$$\varsigma = \zeta + \zeta' \tag{28}$$

and

$$\upsilon = (\zeta - \zeta') = \ln\left[(x_1 - x_{1\nu}) / (x_1' - x_1^{\nu}) \right] \tag{29}$$

Thus, a function which is dependent on ζ and ζ' can be written in terms of a variable υ, which is only a function of the separation distance between the two points in the transformed coordinate, and a variable ς which is dependent on the position of the two points relative to the jet origin. Since the second portion of the hypothesized similarity solution must be independent of the position of the two points relative to the jet origin by definition, it follows that this portion of the similarity solution must be a function of the variable υ only.

The hypothesized forms of the two-point similarity solutions, consistent with the guidelines outlined above, are given by

$$\begin{aligned}
\overline{u_i u_j'}\,(x_1, x_1', x_2, x_2', x_3, x_3') &= P_{i,j}(x_1, x_1')\, q_{i,j}(\eta, \eta', x_3, x_3', \upsilon) \\
\overline{u_i u_j' u_k} &= T^1_{ik,j}(x_1, x_1')\, t^1_{ik,j}(\eta, \eta', x_3, x_3', \upsilon) \\
\overline{u_i u_j' u_k'} &= T^2_{i,jk}(x_1, x_1')\, t^2_{i,jk}(\eta, \eta', x_3, x_3', \upsilon) \\
\overline{p u_j'} &= \pi^1_j(x_1, x_1')\, p^1_j(\eta, \eta', x_3, x_3', \upsilon) \\
\overline{p' u_i} &= \pi^2_i(x_1, x_1')\, p^2_i(\eta, \eta', x_3, x_3', \upsilon)
\end{aligned} \tag{30}$$

The summation convention, where summation is implied over repeated subscripts, is not utilized in these equations nor in any equation hereafter unless explicitly stated. The mixed coordinate notation in equation 30 (i.e. x_1, x_1', and υ) is included to facilitate the comparison between the scales for the two-point correlation tensors and the relevant scales for the single-point moments (since $P_{i,j}$, $T^1_{ki,j}$, and $T^2_{i,kj}$ etc. must be consistent with the scale for the single-point moments in the limit of zero separation distance).

Substituting these similarity solution into the equation 25 yields

$$\begin{aligned}
&\left[U_s \frac{\partial P_{i,j}}{\partial x_1}\right] f q_{i,j} + \left[\frac{P_{i,j} U_s}{x_1}\right]\left\{ f\frac{\partial q_{i,j}}{\partial \upsilon} - \frac{1}{\eta}\left(\frac{\partial q_{i,j}}{\partial \eta} + \frac{q_{3,j}}{\eta}\delta_{i3}\right)\int_0^{\eta} \tilde{\eta} f d\tilde{\eta} + \frac{q_{3,j}}{\eta} f \delta_{i3}\right\} + \left[U_s' \frac{\partial P_{i,j}}{\partial x_1'}\right] f' q_{i,j} - \\
&\left[\frac{P_{i,j} U_s'}{x_1}\right]\left\{ f'\frac{\partial q_{i,j}}{\partial \upsilon} + \frac{1}{\eta'}\left(\frac{\partial q_{i,j}}{\partial \eta'} + \frac{q_{3,j}}{\eta'}\delta_{i3}\right)\int_0^{\eta'} \tilde{\eta} f d\tilde{\eta} - f' q_{3,j}\delta_{i3}\right\} = -\left[\frac{\partial T^1_{1i,j}}{\partial x_1}\right] t^1_{1i,j} - \left[\frac{T^1_{1i,j}}{x_1}\right]\left\{-\eta \frac{\partial t^1_{1i,j}}{\partial \eta} + \frac{\partial t^1_{1i,j}}{\partial \upsilon}\right\} \\
&- \left[\frac{T^1_{2i,j}}{\delta}\right]\frac{1}{\eta}\frac{\partial \eta t^1_{2i,j}}{\partial \eta} - \left[\frac{T^1_{3i,j}}{\delta}\right]\frac{1}{\eta}\frac{\partial t^1_{3i,j}}{\partial \theta} + \left[\frac{T^1_{33,j}}{\delta(x_1)}\right]\frac{t^1_{33,j}}{\eta}\delta_{i2} - \left[\frac{T^1_{23,j}}{\delta}\right]\frac{t^1_{23,j}}{\eta}\delta_{i3} - \left[\frac{\partial T^2_{i,1j}}{\partial x_1'}\right] t^1_{1i,j}
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{T^2_{i,1j}}{x_1'}\right]\left\{\eta'\frac{\partial t^1_{1i,j}}{\partial\eta'}+\frac{\partial t^1_{1i,j}}{\partial\upsilon}\right\}-\left[\frac{T^2_{i,2j}}{\delta'}\right]\frac{1}{\eta'}\frac{\partial\eta t^2_{i,2j}}{\partial\eta'}+\left[\frac{T^2_{i,3j}}{\delta'}\right]\frac{1}{\eta'}\frac{\partial t^2_{i,3j}}{\partial\theta}+\left[\frac{T^2_{i,33}}{\delta'}\right]\frac{t^2_{i,33}}{\eta'}\delta_{j2}-\left[\frac{T^2_{i,23}}{\delta'}\right]\frac{t^2_{i,23}}{\eta}\delta_{j3}\\
&-\frac{1}{\rho}\left\{\left[\frac{\partial\pi^1_j}{\partial x_1}\right]p^1_j+\left[\frac{\pi^1_j}{x_1}\right]\left(-\eta\frac{\partial p^1_j}{\partial\eta}+\frac{\partial p^1_j}{\partial\upsilon}\right)\right\}\delta_{i1}-\frac{1}{\rho}\left[\frac{\pi^1_j}{\delta}\right]\left(\frac{\partial p^1_j}{\partial\eta}\delta_{i2}+\frac{1}{\eta}\frac{\partial p^1_j}{\partial\theta}\delta_{i3}\right)-\\
&\frac{1}{\rho}\left\{\left[\frac{\partial\pi^2_i}{\partial x_1'}\right]p^2_j-\left[\frac{\pi^2_i}{x_1'}\right]\left(\eta'\frac{\partial p^2_i}{\partial\eta'}+\frac{\partial p^2_i}{\partial\upsilon}\right)\right\}\delta_{j1}-\frac{1}{\rho}\left[\frac{\pi^2_i}{\delta'}\right]\left(\frac{\partial p^2_i}{\partial\eta'}\delta_{j2}-\frac{1}{\eta'}\frac{\partial p^2_i}{\partial\theta}\delta_{j3}\right)\\
&+\left\{\left[\frac{P_{1,j}U_s}{x_1}\right]q_{1,j}\left(f+\eta\frac{df}{d\eta}\right)-\left[\frac{P_{2,j}U_s}{\delta}\right]\;q_{2,j}\frac{df}{d\eta}\delta_{II}\right\}\delta_{i1}+\left\{\left[\frac{P_{1,j}U_s}{x_1}\frac{d\delta}{dx_1}\right]q_{1,j}\left(\eta f+\eta^2\frac{df}{d\eta}\right)\right.\\
&\left.-\left[\frac{P_{2,j}U_s}{x_1}\right]q_{2,j}\left(\eta\frac{df}{d\eta}+\left(\;+\frac{1}{\eta^2}\int_0^\eta\tilde\eta f d\tilde\eta\right)\right)\right\}\delta_{i2}+\left\{\left[\frac{P_{i,1}U_s'}{x_1'}\right]q_{i,1}\left(f+\eta'\frac{df}{d\eta'}\right)-\left[\frac{P_{i,2}U_s'}{\delta'}\right]q_{i,2}\frac{df}{d\eta'}\right\}\delta_{j1}\\
&+\left\{\left[\frac{P_{i,1}U_s'}{x_1'}\frac{d\delta}{dx_1}\right]q_{i,1}\left(\eta' f+\eta'^2\frac{df}{d\eta'}\right)-\left[\frac{P_{i,2}U_s'}{x_1'}\right]q_{i,2}\left(\eta'\frac{df}{d\eta'}+\frac{1}{\eta'^2}\int_0^{\eta'}\tilde\eta f d\tilde\eta\right)\right\}\delta_{j2}\\
&+\nu\left\{\left[\frac{\partial^2P_{i,j}}{\partial x_1^2}\right]q_{i,j}+\left[\frac{2P_{i,j}}{x_1^2}-\frac{2}{x_1}\frac{\partial P_{i,j}}{\partial x_1}\right]\left(\eta\frac{\partial q_{i,j}}{d\eta}\right)+\left[\frac{2}{x_1}\frac{\partial P_{i,j}}{\partial x_1}-\frac{P_{i,j}}{x_1^2}\right]\frac{\partial q_{i,j}}{\partial\upsilon}-\left[\frac{2P_{i,j}}{x_1^2}\right]\left(\eta\frac{\partial^2 q_{i,j}}{\partial\eta\partial\upsilon}-\eta^2\frac{\partial^2 q_{i,j}}{\partial\eta^2}-\right.\right.\\
&\left.\frac{\partial^2 q_{i,j}}{\partial\upsilon^2}\right)+\left[\frac{P_{i,j}}{\delta^2}\right]\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}\left\{\eta\frac{\partial q_{i,j}}{\partial\eta}\right\}+\frac{1}{\eta^2}\frac{\partial^2 q_{i,j}}{\partial\theta^2}\right)-\left(\left[\frac{P_{2,j}}{\delta^2}\right]\frac{q_{2,j}}{\eta^2}+\left[\frac{2P_{3,j}}{\delta^2}\right]\frac{1}{\eta^2}\frac{\partial q_{3,j}}{\partial\theta}\right)\delta_{i2}+\\
&\left.+\left(\left[\frac{2P_{2,j}}{\delta^2}\right]\frac{1}{\eta^2}\frac{\partial q_{2,j}}{\partial\theta}-\left[\frac{P_{3,j}}{\delta^2}\right]\frac{q_{3,j}}{\eta^2}\right)\delta_{i3}\right\}+\nu\left\{\left[\frac{\partial^2P_{i,j}}{\partial x_1'^2}\right]q_{i,j}+\left[\frac{2P_{i,j}}{x_1'^2}-\frac{2}{x_1'}\frac{\partial P_{i,j}}{\partial x_1'}\right]\eta'\frac{\partial q_{i,j}}{d\eta'}\right.\\
&-\left[\frac{2}{x_1'}\frac{\partial P_{i,j}}{\partial x_1'}-\frac{P_{i,j}}{x_1'^2}\right]\frac{\partial q_{i,j}}{\partial\upsilon}+\left[\frac{2P_{i,j}}{x_1'^2}\right]\left(\eta'\frac{\partial^2 q_{i,j}}{\partial\eta'\partial\upsilon}+\eta'^2\frac{\partial^2 q_{i,j}}{\partial\eta'^2}+\frac{\partial^2 q_{i,j}}{\partial\upsilon^2}\right)+\left[\frac{P_{i,j}}{\delta'^2}\right]\left(\frac{1}{\eta'}\frac{\partial}{\partial\eta'}\left(\eta'\frac{\partial q_{i,j}}{\partial\eta'}\right)+\frac{1}{\eta'^2}\frac{\partial^2 q_{i,j}}{\partial\theta^2}\right)\\
&\left.-\left(\left[\frac{P_{i,2}}{\delta'^2}\right]\frac{q_{i,2}}{\eta'^2}-\left[\frac{2P_{i,3}}{\delta'^2}\right]\frac{1}{\eta'^2}\frac{\partial q_{i,3}}{\partial\theta}\right)\delta_{j2}-\left(\left[\frac{2P_{i,2}}{\delta'^2}\right]\frac{1}{\eta'^2}\frac{\partial q_{2,j}}{\partial\theta}+\left[\frac{P_{i,3}}{\delta'^2}\right]\frac{q_{i,3}}{\eta^2}\right)\right\}
\end{aligned}
\tag{31}
$$

where U_s' is equal to $U_s(x_1')$, δ' ia equal to $\delta(x_1')$ *etc.*

The portion of each term (in equation 31) dependent on the position of the jet origin is included in square brackets. Equation 31 will admit to similarity solutions of the hypothesized form if it is possible to choose the scale functions ($P_{i,j}$ etc.) in a manner such that all the terms in the square brackets are only a function υ only.

A closer examination of equations 31 reveals that the functions in the square brackets can be easily divided into two groups. The terms in each of these groups are proportional if the terms satisfy constraints analogous to the constraints outlined for the single-point equations. The constraints for the first group are given by

$$U_s \frac{\partial P_{i,j}}{\partial x_1} \sim \frac{P_{i,j} U_s}{x_1} \sim \frac{\partial T^1_{1i,j}}{\partial x_1} \sim \frac{T^1_{1i,j}}{x_1} \sim \frac{T^1_{2i,j}}{\delta} \sim \frac{T^1_{3i,j}}{\delta} \sim \frac{T^1_{33,j}}{\delta}\delta_{i2} \sim \frac{T^1_{23,j}}{\delta}\delta_{i3} \sim \left(\frac{\partial \pi^1_j}{\partial x_1} \sim \frac{\pi^1_j}{x_1}\right)\delta_{i1} \sim \frac{\pi^1_j}{\delta}\delta_{i2} \sim$$
$$\frac{\pi^1_j}{\delta}\delta_{i3} \sim \left(\frac{P_{1,j} U_s}{x_1} \sim \frac{P_{2,j} U_s}{\delta}\right)\delta_{i1} \sim \left(\frac{P_{1,j} U_s}{x_1}\frac{d\delta}{dx_1} \sim \frac{P_{2,j} U_s}{x_1}\right)\delta_{i2} \sim \nu\left\{\frac{\partial^2 P_{i,j}}{\partial x_1^2} \sim \frac{1}{x_1}\frac{\partial P_{i,j}}{\partial x_1} \sim \frac{P_{i,j}}{x_1^2} \sim \frac{P_{i,j}}{\delta^2}\right\}$$
$$-\nu\left(\frac{P_{2,j}}{\delta^2} \sim \frac{P_{j,3}}{\delta^2}\right)\delta_{i2} \sim \nu\left(\frac{P_{j,2}}{\delta^2} \sim \frac{P_{j,3}}{\delta^2}\right)\delta_{j3} \tag{32}$$

The constraints for the second group are given by

$$U_s' \frac{\partial P_{i,j}}{\partial x_1'} \sim \frac{P_{i,j} U_s'}{x_1'} \sim \frac{\partial T^2_{i,j1}}{\partial x_1'} \sim \frac{T^2_{i,j1}}{x_1'} \sim \frac{T^2_{i,j2}}{\delta'} \sim \frac{T^2_{i,j3}}{\delta'} \sim \frac{T^2_{i,33}}{\delta'}\delta_{j2} \sim \frac{T^1_{i,23}}{\delta'}\delta_{j3} \sim \left(\frac{\partial \pi^2_i}{\partial x_1'} \sim \frac{\pi^2_i}{x_1'}\right)\delta_{j1} \sim \frac{\pi^2_i}{\delta'}\delta_{j2} \sim$$
$$\frac{\pi^2_i}{\delta'}\delta_{j3} \sim \left(\frac{P_{i,1} U_s'}{x_1'} \sim \frac{P_{i,2} U_s'}{\delta'}\right)\delta_{j1} \sim \left(\frac{P_{i,1} U'_s}{x_1'}\frac{d\delta}{dx_1} \sim \frac{P_{i,2} U_s'}{x_1'}\right)\delta_{j2} \sim \nu\left\{\frac{\partial^2 P_{i,j}}{\partial x'^2_1} \sim \frac{1}{x_1'}\frac{\partial P_{i,j}}{\partial x_1'} \sim \frac{P_{i,j}}{x'^2_1} \sim \frac{P_{i,j}}{\delta'^2}\right\} \sim$$
$$\nu\left(\frac{P_{i,2}}{\delta'^2} \sim \frac{P_{i,3}}{\delta'^2}\right)\delta_{j2} \sim \nu\left(\frac{P_{i,2}}{\delta'^2} \sim \frac{P_{i,3}}{\delta'^2}\right)\delta_{j3} \tag{33}$$

Examining the first two terms in equations 32 and 33 (and the consistency condition for zero separation) leads to the choice of the scale for the two-point velocity correlation tensor, $P_{i,j}$, given by

$$P_{i,j} \sim U_s(x_1)\, U_s(x_1')\left(\frac{d\delta}{dx_1}\right)^{b(i,j)} \tag{34}$$

where b(i,j) is a numerical power which is a function of the value of i and j. The function b(i,j) must be chosen so the scales for the two-point velocity correlation tensor are consistent with the scales chosen for the single-point moments. However, neither the two-point nor the single-point similarity analyses yield sufficient information to choose the values of b(i.j).

Once the scale for $P_{i,j}$ is chosen, all of the other scales are determined. These are given by

$$T^1_{1i,j} \sim U_s(x_1)\, P_{i,j}, \quad T^1_{2i,j} \sim U_s(x_1)\, P_{i,j}\frac{d\delta}{dx_1}, \quad T^1_{3i,j} \sim U_s(x_1)\, P_{i,j}\frac{d\delta}{dx_1}$$

$$T^2_{i,j1} \sim U_s(x_1')\, P_{i,j}, \quad T^2_{i,j2} \sim U_s(x_1')\, P_{i,j}\frac{d\delta}{dx_1}, \quad T^2_{i,j3} \sim U_s(x_1')\, P_{i,j}\frac{d\delta}{dx_1}$$

$$\pi^1_j \sim U_s(x_1)^2\, U_s(x_1'), \quad \pi^2_i \sim U_s(x_1)\, U_s(x_1')^2 \tag{35}$$

where growth rate terms are included in the some of the scales when the equations indicate they are appropriate.

Note that the viscous terms in the two-point correlation tensor equation consist of a linear operator applied to the two-point velocity correlation tensor. Thus, *one of the key differences between the two-point similarity analysis and the single-point analysis is that the viscous terms in the two-point equations do not introduce additional arbitrary scales*, as they did in the single-point analysis. It is clear that the scales for the viscous terms in equation 31 are only proportional to each other if the growth rate of the jet is constant. In addition, the viscous terms are proportional to the convective terms if

$$\frac{P_{i,j}U_s}{x_1} \sim \frac{P_{i,j}}{x_1^2} \quad \rightarrow U_s x_1 \sim const \tag{36}$$

For this flow (with a constant growth rate), this condition implies that the Reynolds number of the jet based on the similarity variables must be a constant. This requirement is analogous to the condition outlined by George (1994) for a relative balance between the convection and dissipation rate terms in the single-point Reynolds stress equations.

It is straightforward to demonstrate that both groups of constraints in equations 32 and 33 are satisfied when the scales are chosen as defined in equations 34 and 35. In addition, the ratio of the terms from the two groups in equation 32 and 33 is given by

$$\frac{P_{i,j}U_s(x_1)}{x_1}\left(\frac{P_{i,j}U_s(x_1')}{x_1'}\right)^{-1} \sim \frac{x_1'^2}{x_1^2} = e^{-2\upsilon} \tag{37}$$

which is only a function of the separation distance between the points in the transformed coordinate. Therefore, when the scales are chosen as in equations 34 and 35, equation 31 can be written in a form which is independent of the position of the jet origin.

For example, if the scales for $P_{i,j}$ are defined as

$$P_{3,3} = P_{2,2} = U_s(x_1)\,U_s(x_1')\left(\frac{d\delta}{dx_1}\right)^2 = P_{3,2} = P_{2,3} = P_{1,1}\left(\frac{d\delta}{dx_1}\right)^2$$

$$P_{1,2} = P_{2,1} = P_{3,1} = P_{1,3} = U_s(x_1)\,U_s(x_1')\left(\frac{d\delta}{dx_1}\right) \tag{38}$$

and the proportionalities in equation 35 are taken as equalities, equation 31 can be rewritten as

$$e^{-\upsilon}\left[-fq_{i,j} + f\frac{\partial q_{i,j}}{\partial\upsilon} - \frac{1}{\eta}\left(\frac{\partial q_{i,j}}{\partial\eta} + \frac{q_{3,j}}{\eta}\delta_{i3}\right)\int_0^{\eta}\tilde{\eta}fd\tilde{\eta} + fq_{3,j}\delta_{i3}\right] +$$

$$e^{\upsilon}\left[-f'q_{i,j} - f'\frac{\partial q_{i,j}}{\partial\upsilon} - \frac{1}{\eta'}\left(\frac{\partial q_{i,j}}{\partial\eta'} + \frac{q_{3,j}}{\eta'}\delta_{i3}\right)\int_0^{\eta'}\tilde{\eta}fd\tilde{\eta} + f'q_{3,j}\delta_{i3}\right] =$$

$$
\begin{aligned}
& e^{-\upsilon}\left[2t^{1}_{1i,j}+\eta\frac{\partial t^{1}_{1i,j}}{\partial\eta}-\frac{\partial t^{1}_{1i,j}}{\partial\upsilon}-\frac{1}{\eta}\frac{\partial\eta t^{1}_{2i,j}}{\partial\eta}-\frac{1}{\eta}\frac{\partial t^{1}_{3i,j}}{\partial\theta}+\frac{t^{1}_{33,j}}{\eta}\delta_{i2}-\frac{t^{1}_{23,j}}{\eta}\delta_{i3}\right] \\
& +e^{\upsilon}\left[2t^{2}_{i,1j}+\eta'\frac{\partial t^{2}_{i,1j}}{\partial\eta'}+\frac{\partial t^{2}_{i,1j}}{\partial\upsilon}-\frac{1}{\eta'}\frac{\partial\eta t^{2}_{i,2j}}{\partial\eta'}+\frac{1}{\eta'}\frac{\partial t^{1}_{i,3j}}{\partial\theta}+\frac{t^{1}_{i,33}}{\eta'}\delta_{j2}-\frac{t^{1}_{i,32}}{\eta}\delta_{j3}\right] \\
& +\frac{e^{-\upsilon}}{\rho}\left(\frac{d\delta}{dx_1}\right)^{-b}\left\{\left[2p^{1}_{j}+\eta\frac{\partial p^{1}_{j}}{\partial\eta}-\frac{\partial p^{1}_{j}}{\partial\upsilon}\right]\delta_{i1}-\left(\frac{d\delta}{dx_1}\right)^{-1}\left(\frac{\partial p^{1}_{j}}{\partial\eta}\delta_{i2}+\frac{1}{\eta}\frac{\partial p^{1}_{j}}{\partial\theta}\delta_{i3}\right)\right\} \\
& +\frac{e^{\upsilon}}{\rho}\left(\frac{d\delta}{dx_1}\right)^{-b}\left\{\left[2p^{2}_{i}+\eta'\frac{\partial p^{2}_{i}}{\partial\eta'}+\frac{\partial p^{2}_{i}}{\partial\upsilon}\right]\delta_{j1}-\left(\frac{d\delta}{dx_1}\right)^{-1}\left(\frac{\partial p^{2}_{i}}{\partial\eta'}\delta_{j2}-\frac{1}{\eta'}\frac{\partial p^{2}_{i}}{\partial\theta}\delta_{j3}\right)\right\} \\
& +e^{-\upsilon}\left[\left\{q_{1,j}\left(f(\eta)+\eta\frac{df}{d\eta}\right)-q_{2,j}\frac{df}{d\eta}\right\}\delta_{i1}+\left\{q_{1,j}\left(\eta f+\eta^{2}\frac{df}{d\eta}\right)-\right.\right. \\
& \left.\left.q_{2,j}\left(\eta\frac{df}{d\eta}+\frac{1}{\eta^{2}}\int_{0}^{\eta}\tilde{\eta}fd\tilde{\eta}\right)\right\}\delta_{i2}\right]+e^{\upsilon}\left[\left\{q_{i,1}\left(f+\eta'\frac{df}{d\eta'}\right)-q_{i,2}\frac{df}{d\eta'}\right\}\delta_{j1}-\right. \\
& \left.q_{i,2}\left(\eta'\frac{df}{d\eta'}+\frac{1}{\eta'^{2}}\int_{0}^{\eta'}\tilde{\eta}fd\tilde{\eta}\right)\right\}\delta_{j2}\Big]+e^{-\upsilon}\left[\frac{1}{Re_{\delta}}\frac{d\delta}{dx_1}\right]\left\{2q_{i,j}+4\eta\frac{\partial q_{i,j}}{\partial\eta}-\right. \\
& -3\frac{\partial q_{i,j}}{\partial\upsilon}-2\eta\frac{\partial^{2}q_{i,j}}{\partial\eta\partial\upsilon}+\eta^{2}\frac{\partial^{2}q_{i,j}}{\partial\eta^{2}}+\frac{\partial^{2}q_{i,j}}{\partial\upsilon^{2}}+\left(\frac{d\delta}{dx_1}\right)^{-2}\left[\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial q_{i,j}}{\partial\eta}\right)+\frac{1}{\eta^{2}}\frac{\partial^{2}q_{i,j}}{\partial\theta^{2}}\right. \\
& \left.\left.-\left(\frac{q_{2,j}}{\eta^{2}}+\frac{2}{\eta^{2}}\frac{\partial q_{3,j}}{\partial\theta}\right)\delta_{i2}+\left(\frac{2}{\eta^{2}}\frac{\partial q_{2,j}}{\partial\theta}-\frac{q_{3,j}}{\eta^{2}}\right)\delta_{i3}\right]\right\}+e^{\upsilon}\left[\frac{1}{Re_{\delta}}\frac{d\delta}{dx_1}\right]\left\{2q_{i,j}+\right. \\
& 4\eta'\frac{\partial q_{i,j}}{\partial\eta'}+3\frac{\partial q_{i,j}}{\partial\upsilon}+2\eta\frac{\partial^{2}q_{i,j}}{\partial\eta'\partial\upsilon}+\eta'^{2}\frac{\partial^{2}q_{i,j}}{\partial\eta'^{2}}+\frac{\partial^{2}q_{i,j}}{\partial\upsilon^{2}}+\left(\frac{d\delta}{dx_1}\right)^{-2}\left[\frac{1}{\eta'}\frac{\partial}{\partial\eta'}\left(\eta'\frac{\partial q_{i,j}}{\partial\eta'}\right)\right. \\
& \left.\left.+\frac{1}{\eta'^{2}}\frac{\partial^{2}q_{i,j}}{\partial\theta^{2}}-\left(\frac{q_{2,j}}{\eta'^{2}}-\frac{2}{\eta'^{2}}\frac{\partial q_{3,j}}{\partial\theta}\right)\delta_{i2}-\left(\frac{2}{\eta'^{2}}\frac{\partial q_{2,j}}{\partial\theta}+\frac{q_{3,j}}{\eta'^{2}}\right)\delta_{j3}\right]\right\}
\end{aligned}
\tag{39}
$$

If equation 39 has a non-trivial solution then the equations for the two-point velocity correlation tensor admit to a similarity solution. With some additional effort, it is also possible to demonstrate that these solutions are Hermitian and maintain the reflective properties in the azimuthal direction.

The similarity analysis of the two-point equations was carried out using a virtual origin to model the flow exit, so the similarity solution defined by equation 39 must be viewed as an asymptotic state of the turbulence in a jet. However, this asymptotic state may not be unique. Some of the coefficients in equation 39 have an explicit dependence on both the growth rate and the Reynolds number of the jet. There is no obvious technique to eliminate both of these factors from the similarity equation, so the form similarly solution for the two-point correlation tensor may be dependent on source conditions to the extent that the source conditions affect the value of Reynolds number and the growth rate of the jet.

THE PRESSURE FIELD

Since the two-point velocity information is now available in similarity form, it is also possible to examine the pressure field generated by the region described by the similarity solution to determine if this field yields a pressure-velocity correlation which is consistent with the hypothesized similarity form. For an incompressible free shear layer (no boundaries) it is possible to demonstrate that the instantaneous fluctuating pressure field (that is the instantaneous pressure minus the mean pressure) is related to the velocity field in the volume (excluding the singularity at the virtual origin of the flow) by the expression (v. Towsend, 1976 for the expression in cartesian coordinates)

$$\frac{p}{\rho} = -\frac{1}{4\pi}\iiint\left[2\frac{1}{h^{j''}}\frac{\partial U_i''}{\partial x_j''}\frac{1}{h^{i''}}\frac{\partial u_j''}{\partial x_i''} + 2\frac{U_2''u_2''}{x_2''^2} + 2\frac{U_2''}{x_2''^2}\frac{\partial u_3''}{\partial x_3''}\right]$$
$$\frac{x_2''dx_3''dx_2''dx_1''}{(x_2^2 + x_2''^2 - 2x_2x_2''\cos(x_3 - x_3) + (x_1 - x_1'')^2)^{1/2}}$$
$$-\frac{1}{4\pi}\iiint\frac{1}{h^{j''}}\frac{\partial}{\partial x_j''}\frac{1}{h_i''}\frac{\partial}{\partial x_i''}\left(u_i''u_j'' - \overline{u_i''u_j''}\right) + \frac{2}{x_2''}\frac{1}{h^{m''}}\frac{\partial}{\partial x_m''}\left[u_2''u_k'' - \overline{u_2''u_k''}\right] -$$
$$2\frac{\partial}{\partial x_3''}\left(\frac{u_3''u_3''}{x_2''}\right)\frac{x_2''dx_3''dx_2''dx_1''}{(x_2^2 + x_2''^2 - 2x_2x_2''\cos(x_3 - x_3) + (x_1 - x_1'')^2)^{1/2}} \tag{40}$$

where the integration is carried out over the volume excluding the virtual origin (for example using a cut along the $-x_1$ axis). The fluctuating pressure field also includes a contribution from the surface around the singularity. The contribution of the pressure from this volume integration to the pressure-velocity correlation can be determined by multiplying this equation by u_k' and averaging. It is now possible to determine how the flow described by the similarity solution contributes to this pressure velocity correlation. This only includes the flow on the positive x_1 side of the axis excluding the origin (taking the virtual origin at the origin of the x_k coordinate system with the jet flowing in the positive x_1 direction) because the similarity coordinate system defined in both the single and two-point similarity analysis becomes singular at the origin and is incapable of describing the flow behind the virtual origin.

Substituting the hypothesized similarity solution for the mean velocity field. the two-point velocity correlation tensor, and the turbulent transfer terms into the resulting equation yields

$$\frac{\overline{pu_k'}}{\rho} = -\left[U_s^2 U_s'\left(\frac{d\delta}{dx_1}\right)^{a(k)}\right]\left\{\frac{1}{2\pi}\iiint e^{2\upsilon''}\left[\left(f'' + \eta''\frac{df''}{d\eta''}\right)\left(\frac{\partial q_{k,1}''}{\partial\upsilon''} + \eta''\frac{\partial q_{k,1}''}{\partial\eta''} + q_{k,1}''\right) - \frac{df''}{d\eta''}\left(\frac{\partial q_{k,2}''}{\partial\upsilon''} + \eta''\frac{\partial q_{k,2}''}{\partial\eta''} + q_{k,2}''\right) - \left(\eta''f'' + \eta''\frac{d^2f''}{d\eta''^2}\right)\frac{\partial q_{k,1}''}{\partial\eta''} + 2\left(\eta''\frac{df''}{d\eta''} + \frac{F''}{\eta''^2}\right)\frac{\partial q_{k,2}''}{\partial\eta''} + 2\left(-\frac{F''}{\eta''} + \eta''f''\right)\left(\frac{q_{k,2}''}{\eta''^2} - \frac{1}{\eta''^2}\frac{\partial q_{k,3}''}{\partial\theta''}\right)\right]\frac{\eta'' d\theta'' d\eta'' d\upsilon''}{[\eta''^2 + e^{2\upsilon''}\eta^2 - 2e^{\upsilon'}\eta''\eta\cos\theta'' + (d\delta/dx_1)^{-2}(1-e^{\upsilon''})^2]^{1/2}}$$

$$-\frac{1}{4\pi}\iiint e^{2\upsilon''}\left[\left(6 + 3\frac{\partial}{\partial\upsilon''} + \frac{\partial^2}{\partial\upsilon''^2} + 4\eta''\frac{\partial}{\partial\eta''} + \eta''^2\frac{\partial^2}{\partial\eta''^2} + 2\eta''\frac{\partial^2}{\partial\upsilon''\partial\eta''}\right)t_{k,11}'' + \left(\frac{\partial^2}{\partial\eta''^2} - \frac{2}{\eta''}\frac{\partial}{\partial\eta''}\right)t_{k,22}'' - 2\left(3\frac{\partial}{\partial\eta''} - \frac{\partial^2}{\partial\upsilon''\partial\eta''} + \eta''\frac{\partial^2}{\partial\eta''^2} + \frac{2}{\eta''} - \frac{1}{\eta''}\frac{\partial}{\partial\upsilon} + \frac{\partial}{\partial\eta''}\right)t_{k,12}'' + 2\left(\frac{2}{\eta''}\frac{\partial}{\partial\theta''} + \frac{1}{\eta''}\frac{\partial^2}{\partial\upsilon''\partial\theta''} + \frac{\partial^2}{\partial\eta''\partial\theta''}\right)t_{k,13}'' - \frac{1}{\eta''^2}\frac{\partial^2 t_{k,33}''}{\partial\theta''^2} \left(\frac{2}{\eta''}\frac{\partial^2}{\partial\eta''\partial\theta''} + \frac{2}{\eta''^2}\frac{\partial}{\partial\theta''} - \frac{1}{\eta''}\frac{\partial}{\partial\eta''}\right)t_{k,32}''\right.$$

$$\left.\frac{\eta'' d\theta'' d\eta'' d\upsilon''}{[\eta''^2 + e^{2\upsilon''}\eta^2 - 2e^{\upsilon'}\eta''\eta\cos\theta'' + (d\delta/dx_1)^{-2}(1-e^{\upsilon''})^2]^{1/2}}\right\} \tag{41}$$

where a(k) is equal to 1 when k is 1 and it has a value of 2 when k is 2 or 3., F'' is defined as

$$F'' = \int_0^{\eta''}\tilde{\eta} f(\tilde{\eta})\, d\tilde{\eta}, \tag{42}$$

while

$$q_{k,j}''(\upsilon'' - \upsilon, \eta', \eta'', \theta'' - \theta) = q_{k,j}(\zeta' - \zeta'', \eta', \eta'', x_3' - x_3') \tag{43}$$

and

$$t_{k,ij}^2{}''(\upsilon'' - \upsilon, \eta', \eta'', \theta'' - \theta) = t_{k,ij}^2(\zeta' - \zeta'', \eta', \eta'', x_3' - x_3''), \tag{44}$$

where

$$\upsilon'' = \zeta - \zeta'' \tag{45}$$

and

$$\theta'' = x_3 - x_3'' \tag{46}$$

The scale term for the pressure-velocity correlation term in equation 41 is consistent with the scaled required by the similarity analysis. Thus, the similarity hypothesis is internally consistent in the sense that if the motion in the jet is consistent with a two-point similarity hypothesis, then the pressure field generated by the motion in the jet yields a pressure-velocity correlation of a form which also satisfies the similarity hypothesis.

SIMILARITY OF THE VELOCITY GRADIENT MOMENTS

The similarity solution indicates that the two-point velocity correlation tensor can be written in a form which is independent of the position of the two-points relative to the origin of the flow for all separation distances. Hence, it is logical to expect that other moments which can be directly related to the two-point correlation tensor, such the two-point velocity gradient correlation, should have a similarity form when the solution for the two-point velocity correlation tensor can be written in a similarity form.

The dissipation of the kinetic energy per unit mass, ε, can be written as

$$\varepsilon = \lim_{x_i \to x_i'} 2\nu\left(\overline{e_{ij}e_{ij}'}\right) \tag{47}$$

where the e_{ij} is the rate-of-strain tensor. For a cylindrical coordinate system, the rate of strain tensor is given by (v. Batchelor, 1967)

$$e_{ij} = \frac{1}{2}\left(\frac{1}{\mathrm{h}^j}\frac{\partial u_i}{\partial x_j} + \frac{1}{\mathrm{h}^i}\frac{\partial u_j}{\partial x_1}\right) + \frac{u_2}{x_2}\delta_{j3}\delta_{i3} - \frac{1}{2}\frac{u_3}{x_2}[\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}] \tag{48}$$

Therefore, the dissipation of kinetic energy can be written as

$$\varepsilon = \lim_{x_i \to x_i'} 2\nu\left\{\frac{1}{2}\frac{1}{h^j h^{j'}}\frac{\partial^2 R_{i,i}}{\partial x_j \partial x_j'} + \frac{1}{2}\frac{1}{h^i h^{j'}}\frac{\partial^2 R_{j,i}}{\partial x_i \partial x_j'} + \frac{1}{x_2 x_2'}\left(\frac{\partial(R_{3,2} - R_{2,3}/2)}{\partial x_3} + \frac{\partial(R_{2,3} - R_{3,2}/2)}{\partial x_3'}\right) - \frac{1}{2}\left(\frac{1}{x_2}\frac{\partial R_{3,3}}{\partial x_2'} + \frac{1}{x_2'}\frac{\partial R_{3,3}}{\partial x_2}\right) + \frac{R_{2,2}}{x_2 x_2'} + \frac{1}{2}\frac{R_{3,3}}{x_2 x_2'}\right\} \tag{49}$$

where $R_{i,j}$ is the two-point velocity correlation tensor; i.e.,

$$R_{i,j} = \overline{u_i(x_1, x_2, x_3, t)\, u_j(x_1', x_2', x_3', t)} \tag{50}$$

Substituting the similarity solution for the components of the two-point velocity correlation tensor (i.e., equation 31) into the terms in equation 49, it is straightforward to demonstrate that all of the two-point velocity gradient moment can be written as a product of a scale (which are proportional for all of the moments) and a similarity function, so that

$$\varepsilon = 2\nu \lim_{x_i \to x_i'} \left[\frac{U_s(x_1)\, U_s(x_1')}{\delta(x_1)\, \delta(x_1')} \right] D(\eta, \eta', \theta, \upsilon) = \left(\frac{U_s}{\delta} \right)^2 D(\eta) \tag{51}$$

This is consistent with the similarity form for the dissipation required by the similarity analysis of the single-point moment equations since the Reynolds number of the flow is constant.

Using a similar approach it is straight forward the demonstrate that the two-point vorticity correlation can be written in a similarity form when the two-point velocity correlation tensor has a similarity form. In this case, it also follows that the single-point vorticity moments have similarity solutions.

SUMMARY AND CONCLUSION

The review of the similarity hypothesis for the single-point equations illustrates that the results for the axisymmetric jet are analogous to the results derived for the plane jet by George (1994). The component equations for the single-point second order velocity moments admit to a similarity solution only when the growth rate of the jet is linear an the Reynolds number based on the similarity variables is a constant.

The analysis also demonstrates that the equations which govern the evolution of the two-point velocity correlation tensor admit to similarity solutions for the far field of the axisymmetric jet. In this similarity solution the scaled two-point correlation functions are independent of the position of the two points downstream of the jet origin. In flows this case, the single-point similarity solution can be viewed as a special case of the more general two-point similarity solution. The analysis also demonstrated that the Reynolds number of the jet based on the similarity variables must be a constant in order to ensure that the convective and viscous terms in the two-point equations evolve in a manner consistent with the similarity hypothesis for all separation distances.

It was also demonstrated that the two-point velocity-gradient moments have a similarity form when the components of the two-point velocity correlation tensor have a similarity form. This result could be used to demonstrate that the dissipation of the turbulent kinetic energy has a similarity solution which is consistent with form required by the single-point similarity analysis. This result demonstrates that it is reasonable to hypothesize that the dissipation of kinetic energy has a similarity form at a finite Reynolds number when the Reynolds number of the flow is constant as George (1994) did.

At present there is insufficient experimental evidence to validate the two-point similarity hypothesis. An experimental investigation is presently being carried out to generate data for this purpose.

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