Near-Wall Turbulent Flows
R.M.C. So, C.G. Speziale and B.E. Launder (Editors)

Boundary Layers with Pressure Gradient: Another Look at the Equilibrium Boundary Layer

William K. George and Luciano Castillo
Turbulence Research Laboratory
State University of New York
Buffalo, NY 14260

Abstract

The consequences of a full similarity hypothesis for the outer turbulent boundary layer with pressure gradient are explored. Included among them are an outer scaling law for the velocity given by $(U - U_\infty)/U_\infty$ and the fact the $\delta \sim \delta_* \sim \theta$. Matching the outer flow to the near wall region yields power law velocity and friction laws, and determines the outer Reynolds stress scale to be given by u_*^2. These in turn yield the equilibrium condition for the outer flow as $(\delta_*/\rho u_*^2)dP_\infty/dx \equiv \Lambda = constant$, consistent with Clauser's empirical choice.

1 INTRODUCTION

In his now classic paper "*The turbulent boundary layer*", Clauser [1] sets forth in elegant terms the theory of equilibrium turbulent boundary layers. Equilibrium boundary layers satisfy the relation, $(\tilde{\delta}/\rho u_*^2)dP_\infty/dx = constant$ where u_* is the friction velocity and $\tilde{\delta}$ is a length scale identified *only empirically* as the displacement thickness, δ_*. Clauser followed Millikan [2] in analyzing the boundary layer from the perspective of inner and outer scaling laws, and extended the latter to include the pressure gradient parameter identified above. Both inner and outer laws were based on scaling the velocity with u_*; that is, $U/u_* = f(yu_*/\nu)$ and $(U - U_\infty)/u_* = F(y/\delta)$ where δ is an *outer* length scale. Consequences of these choices were the familiar logarithmic velocity profile in the matched layer, and the logarithmic friction law. Less well-known, but of considerable importance in evaluating the correctness of the theory, were that $\delta_*/\delta \sim u_*/U_\infty$ and $\theta/\delta \sim (u_*/U_\infty)(1 + Au_*/U_\infty)$ where θ is the momentum thickness and A is a constant. A consequence of these relations was that the shape factor goes to unity in the limit of infinite Reynolds number; i.e., $H \equiv \delta_*/\theta \to 1$.

That the Millikan/Clauser results are problematical even for the zero pressure gradient boundary layer can be easily seen by approaching the limit as the distance along a surface becomes large (i.e., as $x \to \infty$). The unspecified outer length scale, δ (which by definition must collapse the outer velocity profile) grows without bound relative to the integral thicknesses, δ_* and θ, even though the latter are determined entirely by the outer profiles in the limit. Moreover, δ_* and θ must approach each other since the asymptotic shape factor is unity. In other words, in outer variables there is no boundary layer at all, a very strange boundary layer indeed! [1]

These problems are compounded for the equilibrium boundary layer as proposed by Clauser since, in spite of a theory which argues that δ is the appropriate length scale, experimentally the velocity profiles are observed to collapse with the displacement thickness, δ_* — hence Clauser's choice of $\tilde{\delta} \sim \delta_*$, the theory notwithstanding. On the other hand, the success (disputed by some) of Clauser's theory in accounting for the experimental observations indicates that it might be almost correct, in spite of its internal inconsistencies.

This paper represents an extension of the recent paper by George et al. [3] on the zero pressure gradient boundary layer. The results of the new theory for boundary layers with pressure gradient avoid the inconsistencies of the old, yet rescues many of its attractive features. It can be shown (c.f. George et al. [3]) to have its own unique advantages including power laws instead of logarithms with the result that the integral equations are integrable and the x-dependences obtainable without resort to empirical approximations.

2 The Outer Equations

The outer equations and boundary conditions appropriate to a turbulent boundary layer at high Reynolds number are well-known to be given by

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \frac{\partial}{\partial y}[- < uv >] \tag{1}$$

where $U \to U_\infty$ and $V \to V_\infty$ as $y \to \infty$.

Similarity solutions to equation 1 are sought for which

$$U - U_\infty = U_{so}F(y/\delta) \tag{2}$$

$$-\overline{uv} = R_{so}r_o(y/\delta) \tag{3}$$

where U_{so}, R_{so}, and δ are functions only of x. The two important differences between the present analysis and that of Clauser [1] are immediately apparent:

- The outer velocity scale has not been arbitrarily chosen (as u_* or U_∞) but remains to be determined by the analysis itself.
- The outer Reynolds stress scale has not been arbitrarily chosen (to be u_*^2 or any "velocity squared" form), but remains to be determined from the analysis.

[1] George et al. [3] provide a comprehensive review of the experimental data for the zero pressure gradient boundary layer and the manner in which it has been manipulated incorrectly to support the classical theory.

It is also possible to show from their definitions that the displacement and momentum thicknesses are asymptotically proportional to the boundary layer thickness (i.e. $\delta_* \sim \theta \sim \delta$, George et al. [3]) with an asymptotic shape factor greater than unity.

The free stream flow at the outer edge of the boundary layer and the V-component of the mean velocity in the boundary layer are governed approximately by

$$U_\infty \frac{dU_\infty}{dx} = -\frac{1}{\rho}\frac{dP_\infty}{dx} \tag{4}$$

and

$$V = -\int_0^y \left(\frac{\partial U}{\partial x}\right) dy' \tag{5}$$

Substitution into equation 1 and clearing terms yields

$$\left[\left(\frac{\delta}{U_{so}}\frac{dU_\infty}{dx}\right) + \left(\frac{U_\infty}{U_{so}}\right)\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right] F + \left[\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right] F^2 - \left[\left(\frac{U_\infty}{U_{so}}\frac{d\delta}{dx}\right) + \left(\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right)\right] \overline{y} F'$$
$$- \left[\left(\frac{d\delta}{dx}\right) + \left(\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right)\right] F' \int_0^{\overline{y}} F(\tilde{y}) d\tilde{y} = \left[\frac{R_{so}}{U_{so}^2}\right] r_o' \tag{6}$$

It is clear that full similarity (or self-preservation in the sense of George [5]) is possible only if

$$U_{so} \sim U_\infty \tag{7}$$
$$R_{so} \sim U_\infty^2 \frac{d\delta}{dx} \tag{8}$$
$$\frac{d\delta}{dx} \sim \frac{\delta}{\rho U_\infty}\frac{dU_\infty}{dx} = -\frac{\delta}{\rho U_\infty^2}\frac{dP_\infty}{dx} \tag{9}$$

It is hypothesized that an equilibrium boundary flow in the outer layer will be governed by solutions satisfying the similarity equations above. Since only the free stream velocity and pressure gradient are subject to external control, the boundary layer will adjust itself accordingly. Specifically, the outer profiles of velocity and Reynolds stress will be functions of

$$\overline{y} = y/\delta \tag{10}$$

and

$$\tilde{\Lambda} = \frac{\delta}{\rho U_\infty^2 (d\delta/dx)}\frac{dP_\infty}{dx} = -\frac{\delta}{U_\infty (d\delta/dx)}\frac{dU_\infty}{dx} = constant \tag{11}$$

They may also be expected to exhibit a residual dependence on the local Reynolds number of the flow given by

$$R = \frac{U_\infty \delta}{\nu} \tag{12}$$

Thus the velocity deficit law is given by

$$U - U_\infty = U_\infty F(\overline{y}; \varepsilon, \tilde{\Lambda}) \tag{13}$$

and the Reynolds stress in the outer layer is given by

$$-\overline{uv} = U_{\infty}^{2}(d\delta/dx)r_o(\overline{y};\varepsilon,\tilde{\Lambda}) \tag{14}$$

Before proceeding further, it is worth noting that these conditions were recognized long ago and abandoned in favor of local similarity theories; (e.g. von Karman [4]). The first condition corresponds exactly to the scaling of the outer solution suggested by many from heuristic arguments (c.f. George et al. [3]). Certainly it and the relation between δ and δ_* can not in and of themselves have been the problem since experimentalists have been collapsing boundary layer profiles with U_∞ and δ_* since the beginnings of turbulent boundary layer measurements. It must then have been the condition on the Reynolds stress, equation 7 which presented difficulty to the theoreticians. However, this would have been so only if it were also required (as it was) that $R_{so} = U_{so}^2$, for then it would have also been necessary that $d\delta/dx = constant$. Since the boundary layer was well-known not to grow linearly, von Karman (and many after him as well) was forced to conclude that full self-preservation was not possible, and therefore had to settle for a *locally* self-preserving solution.

George [5], however, has pointed out that there is no reason to insist that $R_{so} = U_{so}^2$, and further argued that this is seldom the case. If this arbitrary requirement is relaxed, then there is no longer the requirement for linear growth, and full similarity of the outer equations becomes tenable. In fact, the outer flow is governed by two velocity scales, U_∞ and a second governing the Reynolds stress satisfying the second condition. What determines this second scale? Clearly it must be determined by the boundary conditions on the Reynolds stress itself. Obviously the homogeneous condition at infinity is of no use, so it must be the conditions on the inside of the boundary layer; namely, those at the outer edge of the wall layer. This matching of inner and outer Reynolds stresses will be carried out below.

3 Similarity of the Inner Equations

The equations for the near wall region of the boundary layer can be integrated to yield

$$u_*^2 = -<uv> + \nu\frac{\partial U}{\partial y} - \frac{y}{\rho}\frac{dP_\infty}{dx} \tag{15}$$

where u_* is the friction velocity and $U \to 0$ at $y = 0$. The presence of u_* in equation 15 does not imply that the wall shear stress is an independent parameter (like ν or U_∞); rather, it is a *dependent* parameter which must be determined by matching solutions of the inner and outer equations.

In keeping with the principle set forth above, similarity solutions to the inner equations are sought of the form

$$U = U_{si}(x)f(y^+) \tag{16}$$

$$-\overline{uv} = R_{si}(x)r_i(y^+) \tag{17}$$

where

$$y^+ \equiv \frac{y}{\eta} \tag{18}$$

and the length scale η remains to be determined.

Substitution into equation 15 and clearing terms yields to leading order

$$\left[\frac{u_*^2}{U_{si}^2}\right] = \left[\frac{R_{si}}{U_{si}^2}\right] r_i + \left[\frac{\nu}{\eta U_{si}}\right] f' - y^+ \left[\frac{\eta}{\rho U_{si}^2}\frac{dP_\infty}{dx}\right] \tag{19}$$

The choice for η is now obviously

$$\eta = \nu / U_{si} \tag{20}$$

from which it follows immediately that similarity solutions are possible only if the inner Reynolds stress scale is given by

$$R_{si} = U_{si}^2 \tag{21}$$

and

$$\frac{\eta}{\rho U_{si}^2}\frac{dP_\infty}{dx} = constant \tag{22}$$

It is easy to show that full similarity of the inner equations is not possible for arbitrary values of dP_∞/dx. If U_{si} is chosen equal to u_*, then equation 19 requires that $(\nu dP_\infty/dx)/\rho u_*^3 = constant$ which is inconsistent with equation 11 unless $dP_\infty/dx = 0$. The other possible choice for the inner scale velocity is $u_p \equiv (\nu dP_\infty/dx/\rho)^{1/3}$ which has the advantage that it satisfies identically equation 22. Unfortunately equation 21 then requires that $u_* \sim u_p$, an unreasonable possibility when $dP_\infty/dx \to 0$. This is, however, the only choice for boundary layers near separation where the wall shear stress becomes vanishingly small, a subject addressed below.

For boundary layers not near separation, it is more convenient to define the inner velocity scale to be the friction velocity which is the only choice for the constant pressure (zero pressure-gradient) boundary layer. This choice will cause no loss of generality as long as the pressure gradient is retained as one of the arguments in r_i. It should be noted that except for the two limiting cases of constant pressure and separation, there is no possibility of a fully similar inner solution and only locally similar solutions can be expected.

If the inner velocity scale is chosen to be

$$U_{si} \equiv u_* \tag{23}$$

then it follows that

$$\eta = \nu / u_* \tag{24}$$

$$R_{si} = u_*^2 \tag{25}$$

These are of course the usual choices for the inner layer, and are the same as those employed by George et al. [3]. The integrated inner equation can now be written (to leading order in ε) as

$$1 = r_i + f' + \lambda y^+ \tag{26}$$

where λ is defined by

$$\lambda \equiv \frac{\nu}{u_*^3}\frac{dP_\infty}{dx} \tag{27}$$

The appropriate scaling laws for the inner boundary layer are given by

$$U = u_* f(y^+;\varepsilon,\lambda) \tag{28}$$
$$-\overline{uv} = u_*^2 r_i(\overline{y};\varepsilon,\lambda) \tag{29}$$

Note that these representations are correct even near separation since the functions depend on both ε and λ. In fact, the necessity of eliminating the dependence on u_* in the limit as $u_* \to 0$ can be used to determine the asymptotic dependence of f and r_i on λ. The results are $f \to \lambda^{1/3} f_s(y^+\lambda^{1/3},\varepsilon)$ and $r_i \to \lambda^{2/3} r_{is}(y^+\lambda^{1/3},\varepsilon)$ in the limit as $\lambda \to \infty$. This means, of course, that u_p is the appropriate scale velocity in the this limit, as noted above.

It will be shown below that matching the inner and outer Reynolds stresses (to first order) requires that $R_{so} \sim R_{si} \sim u_*^2$. Anticipating this, λ can be expressed in terms of the outer pressure parameter Λ as

$$\lambda = \varepsilon\Lambda \tag{30}$$

where Λ is defined as

$$\Lambda = \frac{\delta}{\rho u_*^2}\frac{dP_\infty}{dx} \tag{31}$$

It is useful to rewrite equation 26 in mixed variables using equations 30 and 31 and $\overline{y} = \varepsilon y^+$. The result is

$$1 = r_i + f' + \Lambda\overline{y} \tag{32}$$

Recall that $\overline{y}$ is small in the inner layer ($\overline{y} < 0.1$ is a reasonable approximation to the extent of the inner region). Thus if Λ is small enough so that $\Lambda\overline{y} << 1$, then in this limit, the last term of equation 26 vanishes, and the inner equation is independent of the pressure gradient (to first order).

The reason for using $U_{si} = u_*$ instead of u_p should now be clear since the inner equations reduce to those for zero pressure gradient for moderate values of Λ. On the other hand, if $\Lambda \to \infty$ (as near separation where $u_* \to 0$), then the last term of equation 26 increases without bound and the entire scaling with u_* must be reconsidered. This case will be considered separately below; however, it has already been noted that scaling with u_p is the appropriate choice in this limit.

4 The Velocity in the Matched Layer

The purpose of this section is to effect a matching of the Law of the Wall given by equation 28 to the new Deficit Law set forth in equation 13. There are several ways the matching can be accomplished, all of which yield asymptotically the same results. The procedure utilized here is the same utilized by George et al. [3] which has the advantage that it yields results which are valid at finite Reynolds number, unlike the usual asymptotic matching which is strictly valid only at infinite Reynolds number.

Because of the presence of different velocity scales in the inner and outer solutions, u_* and U_∞ respectively, it is easier to match $(dU/dy)/U$ than the velocity derivative alone, as is usually done. By using the same procedures as George et al. [3] [2] it can be shown that

$$1 + F(\overline{y}, \varepsilon, \Lambda) = C_o(\varepsilon, \Lambda)\overline{y}^{\gamma(\varepsilon,\Lambda)} + B_o(\varepsilon, \Lambda) \quad (33)$$

$$f(y^+, \varepsilon, \Lambda) = C_i(\varepsilon, \Lambda) y^{+\gamma(\varepsilon,\Lambda)} + B_i(\varepsilon, \Lambda) \quad (34)$$

where it is required that

$$\frac{B_o}{B_i} = \frac{u_*}{U_\infty} \quad (35)$$

Thus the velocity profile in the matched layer is a power law with coefficients and exponent which depend in principle on both Reynolds number, $\varepsilon^{-1} = u_*\delta/\nu$, and the pressure gradient parameter Λ. (Note that Λ has been used to replace $\tilde{\Lambda}$ in the outer profile.) Since equations 33 and 34 must be asymptotically independent of Reynolds number, the coefficients and exponent must be asymptotically dependent only on Λ; i.e.

$$\begin{aligned} \gamma(\varepsilon) &\rightarrow \gamma_\infty(\Lambda) \\ B_o(\varepsilon, \Lambda) &\rightarrow 0 \\ B_i(\varepsilon, \Lambda) &\rightarrow B_{i\infty}(\Lambda) \\ C_o(\varepsilon, \Lambda) &\rightarrow C_{o\infty}(\Lambda) \\ C_i(\varepsilon, \Lambda) &\rightarrow C_{i\infty}(\Lambda) \end{aligned}$$

as $\varepsilon \rightarrow 0$. Note that the condition on B_o follows from equation 35 and the fact $u_*/U_\infty \rightarrow 0$ in the limit.

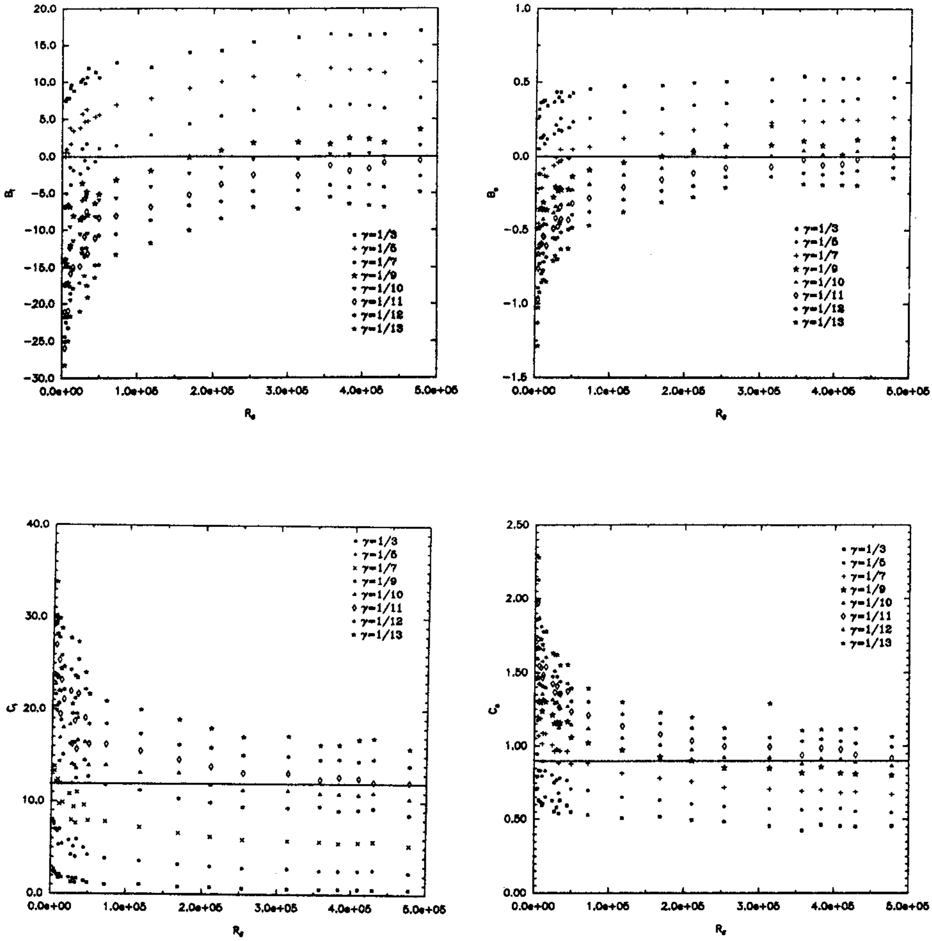
Figures 1 – 4 were obtained by plotting the zero pressure gradient data of Purtell and Klebanoff and Smith and Walker (v. ref [3] for details) as U versus y^γ and fitting the best straight line. The preferred values must be consistent with $B_o \rightarrow 0$ so that the recommended value of γ_∞ is 1/11. The corresponding asymptotic values for the other functions are $B_{i\infty} = 0$, $C_o = 0.9$, and $C_i = 12$. In the absence of other criteria, $\gamma = 1/11$ could be used for all Reynolds numbers and the remaining values chosen from the graph to be functions of the Reynolds number. However these choices are made, these zero pressure gradient values are not appropriate for values of $\Lambda > 5$. Above this value, the parameters retain a dependence on Lambda until the separation limit of $\gamma = 1/2$ is reached.

5 The Friction Law

The requirement that the velocities themselves must match yields a friction law which is valid both asymptotically and for finite Reynolds number. Substituting equations 33 and 34 into equations 28 and 13, and equating velocities yields

$$\frac{u_*}{U_\infty} = \frac{C_o}{C_i}\left(\frac{\eta}{\delta}\right)^\gamma \quad (36)$$

[2]Note that the additive constants B_i and B_o were mistakenly omitted in this earlier paper.



Figures 1 - 4: Reynolds number dependence of B_i, B_o, C_i and C_o for various values of gamma.

It follows immediately that

$$\frac{u_*}{U_\infty} = \left(\frac{C_o}{C_i}\right)^{1/1+\gamma} \left(\frac{U_\infty \delta}{\nu}\right)^{-\gamma/(1+\gamma)} \tag{37}$$

Thus the friction law is also given by a power law, a considerably more convenient form than the implicit logarithmic relation of the old theory. Note that in principle, all of the constants retain a dependence on both ε and Λ.

6 The Reynolds Stress in the Matched Layer

The Reynolds stress can be similarly matched to yield

$$r_o(\overline{y}; \varepsilon, \Lambda) = D_o(\varepsilon, \Lambda)\overline{y}^\beta(\varepsilon, \Lambda) + E_o(\varepsilon, \Lambda) \tag{38}$$
$$r_i(y^+; \varepsilon, \Lambda = D_i(\varepsilon, \Lambda)^\beta(\varepsilon, \Lambda) + E_i(\varepsilon, \Lambda) \tag{39}$$

where a solution is possible only if

$$\frac{D_i}{D_o} = \varepsilon^\beta \frac{E_i}{E_o} \tag{40}$$

and

$$\frac{E_i}{E_o} = \frac{R_{so}}{R_{si}} \tag{41}$$

The Reynolds stress in the matched layer is given (to first order) by

$$r_o = \frac{u_*^2}{R_{so}} + \left[\frac{u_*^2}{R_{so}}\right] \Lambda \overline{y} \tag{42}$$

This can be satisfied by equation 38 only if $\beta = 1$ and

$$E_o = \frac{u_*^2}{R_{so}} \tag{43}$$
$$\frac{D_o}{E_o} = \Lambda \tag{44}$$

In the infinite Reynolds number limit, D_o and E_o become Reynolds number independent so that the similarity condition of equation 11 is indeed $\Lambda = constant$ as stated earlier. This is, of course, Clauser's condition for equilibrium boundary layers as originally derived. However, since δ_*/δ is also asymptotically constant (as demonstrated by George et al. [3], there is no inconsistency between theory and experiment as with the Clauser [1] theory. Equation 43 also makes it clear that the outer Reynolds stress scales with u_*^2 as suggested earlier.

The functions for the inner form of the Reynolds stress in the matched layer can be similarly determined using equations 26 and 39:

$$E_i = 1 \tag{45}$$
$$D_i = -\varepsilon\Lambda \tag{46}$$

7 Summary and Conclusions

By invoking the principle that the outer boundary layer should be described by solutions which were fully similar (as opposed to only locally similar) and by matching these solutions to the wall layer, it has been possible to derive theoretically Clauser's equilibrium boundary layer condition. There are other implications of the theory which page restrictions prohibit discussing here. It was possible to show, however, that the velocity deficit should scale with U_∞ and that the matched layer profiles and friction law were described by power laws. Other consequences of the theory will be presented in subsequent papers. Included among them will be the relation between shape factor and Λ, discussion of the behavior of a boundary layer near separation and a criterion for when separation occurs.

It is of interest to ask when such equilibrium boundary layers exist? From the perspective of this paper it is probably more important to ask when they do *not*, especially in view of the abundant evidence that they do It is suggested that they will *not* only when the time scale over which the external flow or wall conditions (like roughness) change is small compared to the appropriate time scale of the turbulence. Even then the flow will try to settle into a new equilibrium with $\Lambda = constant$, but a different one. Some insight into how this adjustment may occur is provided by the momentum integral equation, all of the terms of which are proportional to one another for equilibrium, but readjust their relative values when the equilibrium is disturbed.

8 Acknowledgements

The authors gratefully acknowledge the interest and insight offered by J. Citriniti, D. Ewing, C.S. Liu, and D. Taulbee in many helpful discussions.

References

[1] Clauser, F.H. The turbulent boundary layer, *Adv. Appl. Mech.*, IV, 1-51 (1954).

[2] Millikan, C.M. A critical discussion of turbulent flows in channels and circular tubes, *Proc. 5th Int. Congr. Appl. Mech.*, 386 – 392, Wiley, NY (1938).

[3] George, W.K., Knecht, P. and Castillo, L. The Zero Pressure Gradient Boundary Layer Revisited, 13th Symp. on Turbulence, Rolla, Mo., Sept 21 – 23 (1992).

[4] von Karman, T. Mechanishe Aehnlichkeit und Turbulenz, *Nachr. Ges. Wiss.*, Goettingen, pp 68 (1930).

[5] George, W.K. Self-Preservation of Turbulent Flows and Its Relation to Initial Conditions and Coherent Structures, in *Advances in Turbulence* (George and Arndt, eds.), 39 – 73, Hemisphere, NY (1989).