THE ZERO-PRESSURE GRADIENT TURBULENT BOUNDARY LAYER REVISITED

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September 1, 1992

1 Introduction

There are few problems in turbulence which are more generally regarded as solved than the scaling laws for the zero pressure gradient boundary layer. The analysis of Millikan [?] which matched inner and outer scaling laws (the Law of the Wall and the Velocity Deficit Law) to obtain logarithmic velocity and friction laws is widely considered to be classical. The acceptance of Millikan's arguments was facilitated by the success of Clauser [?], Hama [?], Coles [?] and others in extending Millikan's arguments to boundary layers with pressure gradients, roughness and compressibility. Equally important was the apparent agreement of experimental data with the theoretical results.

That there have been few dissenters [1] is somewhat surprising in view of some of the unsatisfying features of the Millikan theory. Among them:

1. The velocity profile disappears in the limit of infinite Reynolds number (i.e., $U/U_o = 1$);

2. The outer length scale is not proportional to an integral length scale, and in fact blows up relative to them as the Reynolds number becomes infinite (i.e., δ/δ_* and $\delta/\theta \to \infty$); and

3. The shape factor approaches unity in the infinite Reynolds number limit (i.e., $H = \delta^*/\theta \to 1$).

[1] Perhaps the editor's note to the paper of Long and Chen [?] gives a clue that dissent has simply been suppressed by not publishing it.

In addition, because of the analytical problems presented by the logarithms, only empirical models are possible for the streamwise development of the boundary layer parameters.

In spite of the fact that no shape factors below about 1.25 have been reported, and that boundary layers profiles seem to collapse as well with momentum and displacement thicknesses as with the boundary layer thickness determined from the profile (e.g., $\delta_{0.99}$), it has somehow been possible to live with these 'problems'. Clauser [?] and Tennekes and Lumley [?], for example, use the displacement thickness as the outer length scale in the analysis of equilibrium boundary layers, even though its use is inconsistent with the Millikan analysis (see *iii above*). Acceptance of these ambiguities can in part be understood because of the relatively limited range of Reynolds numbers at which experiments have been performed, but in larger part it probably should be attributed to the absence of rational alternative theories.

This paper reconsiders the fundamental basis of Millikan's analysis. It examines the consequences of an alternative formulation of the outer scaling law which when matched to the Law of the Wall leads to velocity and friction laws which are power laws. Moreover, the new theory removes some of the troubling aspects of the earlier theory; in particular, the outer length scale can be identified with either the momentum or the displacement thickness, and the asymptotic shape factor is greater than unity. Extensions of the theory proposed here to boundary layers with pressure gradients, roughness and compressibility will be reported in subsequent publications.

Part I
Theoretical Considerations

2 Governing Equations and Boundary Conditions

The equations of motion and boundary conditions appropriate to a zero- pressure gradient turbulent boundary layer at high Reynolds number are well-known to be given by

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left[- < uv > +\nu\frac{\partial U}{\partial y}\right] \tag{1}$$

where $U \to U_\infty$ as $y \to \infty$ and $U = 0$ at $y = 0$.

The presence of the no-slip condition precludes the possibility of fully self-preserving solutions, and so locally self-preserving solutions are sought which asymptotically (at infinite Reynolds number) satisfy the following outer and inner equations and boundary conditions:

- Outer Region (Infinite Reynolds Number)

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}[- < uv >] \tag{2}$$

  where $U \to U_\infty$ as $y \to \infty$.

- Inner (or near wall) region (Infinite Reynolds Number)

$$0 = \frac{\partial}{\partial y}\left[- < uv > + \nu\frac{\partial U}{\partial y}\right] \tag{3}$$

  where $U \to 0$ at $y = 0$.

Equation 3 for the inner region can be integrated directly to obtain

$$- < uv > + \nu\frac{\partial U}{\partial y} = \frac{\tau_W}{\rho} \equiv u_*^2 \tag{4}$$

where τ_W is the wall shear stress and u_* is the corresponding friction velocity defined from it. It is clear that in the limit of infinite Reynolds number (but only in this limit) that the total stress is constant across the inner layer, and hence its name "the Constant Stress Layer".

It should be noted that the appearance of u_* in equation 4 does not imply that the wall shear stress is an independent parameter (like ν or U_∞). It enters the problem only because it measures the forcing of the inner flow by the outer, or alternatively, it can be viewed as measuring the retarding effect of the inner flow on the outer. Thus u_* is a *dependent* parameter which must be determined by matching the solutions to the inner and outer equations.

It is also interesting to note that the important feature of the inner layer is that it occurs only because of the necessity of including viscosity in the problem so that the no-slip condition can be met. The outer layer, on the other hand, is dominated by inertia and the effects of viscosity enter only through the matching to the inner layer. In view of these characteristics it would seem reasonable to expect that a matched or overlap layer between the inner and outer layers should include both viscosity and inertia effects, and not be independent of either. Long and Chen [?] have used this observation to argue that the matched layer cannot depend on u_* and y alone as argued in the usual analyses (eg. Tennekes and Lumley [?], Millikan [?], von Karman [?]). This question will be addressed later when comparing the theory proposed below to these classical analyses.

3 The Velocity and Reynolds Stress Profiles

Solutions to the governing equations are sought which depend only on the streamwise coordinate through a local length scale $\delta(x)$. Thus, for the mean velocity

$$U = U(y, \delta, U_\infty, u_*, \nu) \tag{5}$$

It has been previously noted that u_τ cannot be viewed as an independent parameter since it is determined once U_∞, ν and x (or δ) are given. This dependence can be expressed from dimensional considerations as a friction law by

$$\frac{u_\tau}{U_\infty} = \sqrt{\frac{2}{c_f}} = f\left(\frac{u_\tau \delta}{\nu}\right) \tag{6}$$

where

$$c_f \equiv \frac{\tau_W}{\frac{1}{2}\rho U_\infty^2} = 2\frac{u_*^2}{U_\infty^2} \tag{7}$$

Application of the Buckingham Pi theorem to the velocity itself yields a number of possibilities, *all of which describe the variation of the velocity across the entire boundary layer.* Among them are:

$$\frac{U}{u_*} = F_1\left[\frac{yu_*}{\nu}, \frac{u_*\delta}{\nu} or \frac{u_*}{U_\infty}\right] \tag{8}$$

$$\frac{U - U_\infty}{U_\infty} = F_2\left[\frac{y}{\delta}, \frac{u_*\delta}{\nu} or \frac{u_*}{U_\infty}\right] \tag{9}$$

$$\frac{U - U_\infty}{u_*} = F_3\left[\frac{y}{\delta}, \frac{u_*\delta}{\nu} or \frac{u_*}{U_\infty}\right] \tag{10}$$

Note that since u_*/U_∞ and $u_*\delta/\nu$ are related by equation 6, either can be retained (and the other omitted) in equations 8 — **??** with no loss of information. [2]

In the limit as $u_*\delta/\nu \to \infty$ (or $u_*/U_\infty \to 0$), equation 8 becomes asymptotically independent of δ and U_∞, and thus can at most describe a limited region very close to the wall, i.e.,

$$\frac{U}{u_*} = F_{1\infty}\left[\frac{yu_*}{\nu}\right] \tag{11}$$

This is, of course, the familiar *Law of the Wall* expressed in inner variables as originally proposed by Prandtl [**?**].

A similar limiting argument for F_2 and F_3 yields two quite different candidates for an outer profile; namely,

$$\frac{U - U_\infty}{U_\infty} = F_{2\infty}\left(\frac{y}{\delta}\right) \tag{12}$$

and

$$\frac{U - U_\infty}{u_*} = F_{3\infty}\left(\frac{y}{\delta}\right) \tag{13}$$

[2] This fact seems to have escaped Monin and Yaglom [**?**] who dismiss a separate dependence on u_*/U_∞ only on experimental grounds.

The second form given by equation 13 is the traditional choice (originally used by Stanton [?] for pipe flows and adopted by von Karman [?] to the boundary layer. When matched with the inner layer it leads to the familiar logarithmic profiles for the matched layer (Millikan [?]). The first form has only been fleetingly considered by the fluid dynamics community, and discarded in favor of the second alternative. Millikan, for example, appears to have considered it briefly, noted that it leads to self-preserving power law solutions of the outer equations, and then dismissed these solutions as interpolation formulas. Clauser [?] (see also Hinze [?]) plotted only the highest and lowest Reynolds number data of Schultz-Grunow [?] in deficit form as in equation 12, and concluded tha the collapse was not as satisfactory as that obtained using the deficit form of equation 13. There is no evidence that either of these arguments has been refuted, or even questioned, before now. Thus, up until now, the formulation of equation 13 for the outer deficit, and the logarithmic profiles it leads to, have been accepted without question.

The question as to whether the data preclude the outer form of equation 12 can best be addressed by examining it. Figure (1) shows the data of Schultz-Grunow **??** in inner variables corresponding to equation 11. Note particularly the collapse near the wall, and that the point of departure from this single "asymptotic" curve moves away from the wall as the Reynolds number increases. This is, of course, the expected result for an 'inner' scaling law. The opposite behavior appears in Figure 2 which plots the same data in the outer variables of equation 12. Here the collapse is excellent far away from the wall at all Reynolds numbers, and the point of departure from the "asymptote" moves closer to the wall with increasing Reynolds numbers. This is again the expected behavior for an asymptotic outer solution. (Note that Hinze [?] like Clauser [?] before him plotted only the highest and lowest Reynolds numbers, hence the different interpretation). Finally consider the data plotted in the usual outer variables as shown in Figure (3). Unlike the above, the collapse is reasonable at all distances from the wall. Absent almost completely is the expected splitting off with increasing Reynolds number exhibited by both the above plots. While the quality of the collapse of Figure (3) is striking, [3] it is not consistent with the nature of inner and outer scaling laws. Thus, contrary to previous interpretations, the data would seem to indicate a preference for the alternate formulation of the velocity deficit law (equation 12), at least from this perspective.

In view of the above, it is useful to examine whether and why u_* should be a scaling parameter for the outer flow at all, since it is in fact evaluated at the wall. First note that it is only in the limit of infinite Reynolds number where the inner layer is truly a constant stress layer. Thus, only in this limit is the shear stress experienced by the outer flow exactly measured by u_*^2. At all finite Reynolds numbers it only approximately measures the effect of the inner

[3] The reasons for this good collapse may be due to the moderate range of Reynolds numbers of the data, and the mixed (inner and outer) nature of the variables used.

layer on the outer, While the use of u_* as an outer scaling parameter may give reasonable results over a rather large range of Reynolds numbers, it can not be an appropriate choice for the cornerstone of an asymptotic analysis of the outer boundary layer. This can be contrasted with fully-developed turbulent pipe or channel flow where overall balance between pressure and viscous forces on a section of the flow dictate that the outer flow scale with u_*. (The channel flow is discussed in more detail in the Appendix.) An obvious consequence of these observations is that the wall layers of these flows are fundamentally different from the inner boundary layer, contrary to popular belief (c.f. Monin and Yaglom citeml).

In the remainder of this paper the consequences of the alternative formulation of the outer profile given by equation 12 will be explored in detail. Before proceeding, however, the objections to it stated above need to be reexamined. The observation of Millikan [?] that when the deficit is scaled with U_∞ instead of u_* leads to full self-preservation appears to result from the assumption that the scale for the Reynolds stress is equal to the velocity scale squared. As pointed out by George [?], these kinds of assumptions are rarely justified a priori in similarity analyses, and certainly cannot be true here, except in the limit of infinite Reynolds number. More will be said about this later.

4 The Matched Layer: The Traditional View

Millikan [?] matched the 'inner' scaling of equation 11 and the 'outer' scaling of equation 13 in the limit of infinite Reynolds number to obtain the familiar inertial sublayer profiles as

$$\frac{U}{u_*} = \frac{1}{\kappa} ln\left(\frac{y}{\eta}\right) + B \tag{14}$$

$$\frac{U - U_\infty}{u_*} = \frac{1}{\kappa} ln\left(\frac{y}{\delta}\right) + B_1 \tag{15}$$

and a friction law given by

$$\frac{U_\infty}{u_*} = \sqrt{\frac{2}{c_f}} = \frac{1}{\kappa} ln\left(\frac{\delta}{\eta}\right) + (B - B_1) \tag{16}$$

where κ, B, and B_1 are presumed to be universal constants. Note that the last equation implies that $u_*/U_\infty \to 0$ as $\delta/\eta \to \infty$, or equivalently, $c_f \to 0$.

By substituting the inner and outer scaling laws into the defining integrals for the displacement and momentum thicknesses, it follows that

$$\frac{\delta_*}{\delta} = A_1 \sqrt{\frac{c_f}{2}} \tag{17}$$

$$\frac{\theta}{\delta} = A_1\sqrt{\frac{c_f}{2}}\left[1 - A_2\sqrt{\frac{c_f}{2}}\right] \tag{18}$$

where A_1 and A_2 are universal constants which can be evaluated from integrals of the velocity profile. These relations were first given by Clauser [?], and from them he deduced that the shape factor is given by the asymptotic relation

$$H = \frac{\delta_*}{\theta} = 1 - A_2\sqrt{\frac{c_f}{2}} \tag{19}$$

Thus as $\delta/\eta \to 0$ and $c_f \to 0$, $H \to 1$.

The underlying assumption of the above matching is that the inner and outer scaling laws used for the profiles, in fact, have a region of common validity (or overlap) in the limit as $\delta/\eta \to 0$ or $u_*/U_\infty \to 0$. Long and Chen [?] have remarked that it is strange that the matched layer between one characterized by inertia and another characterized by viscosity does not depend on both inertia and viscosity. but only the inertia (hence the term 'inertial sublayer', Tennekes and Lumley [?]). They further suggested that this might be a consequence of improperly matching two layers which did not overlap. The fact that the limiting ratio of the outer length scale δ to both of the commonly used integral length scales, δ_* and θ, is infinite lends considerable weight to their concern. In particular, this implies that from the perspective of the outer flow, the boundary layer does not exist at all in the limit of infinite Reynolds number. If one imagines approaching this limit along a semi-infinite plate where the boundary layer continues to grow, the outer length scale increases faster than any dynamically significant integral length. These is particularly troubling since δ itself is unspecified by the theory and can not be related to physically measurable length scales except through the degenerate expressions above.

5 The Matched Layer: An Revised View

An alternative to the above is to consider matching the 'inner' scaling law (equation 11) with the 'outer' scaling law which uses U_∞ instead of u_* (equation 12). If it is argued that the velocity derivatives given by both 'inner' and 'outer' laws must be the same in the limit of infinite Reynolds number, then for values of y in the overlap region,

$$\lim_{\delta/\eta\to\infty}\left[\frac{u_*}{\eta}F_1'\left(\frac{y}{\eta},\frac{\delta}{\eta}\right) - \frac{U_\infty}{\delta}F_2'\left(\frac{y}{\delta},\frac{\delta}{\eta}\right)\right] \to 0 \tag{20}$$

where $'$ denotes differentiation with respect to the first argument. This condition can be satisfied only if

$$\frac{1}{\eta}F_1'\left(\frac{y}{\eta},\frac{\delta}{\eta}\right) = \frac{1}{\delta}\frac{U_\infty}{u_*}F_2'\left(\frac{y}{\delta},\frac{\delta}{\eta}\right) \tag{21}$$

in the limit as $\delta/\eta \to \infty$ for values of y in the matched layer.

A similar requirement that the velocities themselves must match yields

$$\lim_{\delta/\eta\to\infty} \left[u_* F_1\left(\frac{y}{\eta}, \frac{\delta}{\eta}\right) - U_\infty \left[1 + F_2\left(\frac{y}{\delta}, \frac{\delta}{\eta}\right)\right]\right] \to 0 \tag{22}$$

It is clear that a relation between u_* and U_∞ is required to proceed further. Fortunately such a relation is provided by the form of the friction law, equation 6, which is of the form,

$$\frac{u_*}{U_\infty} = f\left(\frac{\delta}{\eta}\right) \tag{23}$$

The easiest way to proceed is to assume a trial relation for the unknown function f in equation 23, substitute it, and confirm *a posteriori* that the assumed form was correct. Therefore, it will be assumed (subject to confirmation) that

$$\frac{u_*}{U_\infty} = B\left(\frac{\eta}{\delta}\right)^\gamma \tag{24}$$

where B is a constant and γ is a constant exponent, both to be determined later. It follows immediately by substitution into equation 21 and multiplying both sides by $y^{1-\gamma}$ that the matching condition on the velocity derivative becomes

$$B\left(\frac{y}{\delta}\right)^{1-\gamma} F_1'\left(\frac{y}{\eta}, \frac{\delta}{\eta}\right) = \left(\frac{y}{\delta}\right)^{1-\gamma} F_2'\left(\frac{y}{\delta}, \frac{\delta}{\eta}\right) \tag{25}$$

If an inner variable y^+ is defined as

$$y^+ \equiv \frac{y u_*}{\nu} \tag{26}$$

and $\tilde{y}$ is the outer variable defined as

$$\tilde{y} \equiv \frac{y}{\delta} \tag{27}$$

equation 25 can be rewritten as

$$B y^{+1-\gamma} F_1'\left(y^+, \frac{\delta}{\eta}\right) = \tilde{y}^{1-\gamma} F_2'\left(\tilde{y}, \frac{\delta}{\eta}\right) \tag{28}$$

In the limit as $\delta/\eta \to 0$, the ratio of y^+ to $\tilde{y}$ becomes undefined. In this limit, the two sides of equation 28 are functions of different independent variables. Therefore, the matching condition can be satisfied only if both sides equal a constant. For convenience, the constant is chosen to be $C/(\gamma - 1)$ so that

$$B y{+}^{1-\gamma} F_{1\infty}'(y+) = \frac{C}{\gamma - 1} \tag{29}$$

and

$$\tilde{y}^{1-\gamma} F'_{2\infty}(\tilde{y}) = \frac{C}{\gamma - 1} \tag{30}$$

Equations 29 and 30 can be integrated directly to yield

$$BF_{1\infty}(y^+) = Cy^{+\gamma} + C_i \tag{31}$$

and

$$F_{2\infty} = C\tilde{y}^{\gamma} + C_o \tag{32}$$

where C_i and C_o are the integration constants. These can be determined by imposing the requirement that the velocities themselves match in this limit. Substituting equations 31 and 32 into equation 21 and using equation **??** yields

$$C_i \left(\frac{\delta}{\eta}\right)^{-\gamma} = 1 + C_o \tag{33}$$

This can be satisfied for all δ/η only if

$$C_i = 0 \tag{34}$$

and

$$C_o = -1 \tag{35}$$

The inner and outer forms of the velocity profile *in the matched layer* are therefore given by

Inner:

$$\frac{U}{u_*} = B^{-1} C \left(\frac{y}{\eta}\right)^{\gamma} \tag{36}$$

Outer:

$$\frac{U - U_\infty}{U_\infty} = C \left(\frac{y}{\delta}\right)^{\gamma} - 1 \tag{37}$$

or equivalently,

$$\frac{U}{U_\infty} = C \left(\frac{y}{\delta}\right)^{\gamma} \tag{38}$$

It is easy to see by taking the ratio of equations 36 and 38 that the form of the friction law assumed in equation **??** is recovered from the inner and outer matched layer solutions. Thus the velocity profile in the matched layer is represented by a power law with universal constants C and power γ. Note that unlike the former empirical attempts to represent the velocity profile by a power law, here only the matched layer follows this description. Note also that the exponent γ is a universal constant and is not a function of the Reynolds number, at least in the limit of infinite Reynolds number. The power law profile for the matched layer derived above should not be confused with previous attempts to use empirical power law fits to fit the entire profile instead of just the matched layer (e.g. Hinze **??**, Schlicting [**?**]).

There are several interesting points to be made before leaving this section. First, note that (as for the Millikan theory) there are only three constants to be determined from experiment (or other considerations): γ, B, and C. Thus, for example, if plots of the velocity in inner and outer variables are available, the friction law is completely determined. Second, the velocity gradient in the matched layer is never a function of just u_* and y alone (unlike the Millikan solution). This means that the von Karman eddy viscosity with a linear variation of distance from the wall is not consistent (with the theory proposed here. It also means that the theory proposed here overcomes the objection of Long and Chen mentioned earlier since the matched layer retain a dependence on both inertial and viscosity effects. Barenblatt [?] has advanced similar arguments from the perspective of intermediate asymptotics, and also argues on heuristic grounds that the matched layer should be of power law form. In the following section, the further implications of the power law results obtained here on the other parameters of interest will be examined.

6 A Composite Velocity Profile

It is possible to use the information obtained in the preceding section to form a composite velocity profile which is valid over the entire boundary layer. This is accomplished by expressing the inner profile in outer variables, adding it to the outer profile and subtracting the common part (Van Dyke [?]). (Alternatively, the outer profile could be expressed in inner variables, etc.) Since the overlap region provides the common part, the composite velocity profile in outer variables is given by

$$\frac{U}{U_\infty} = 1 + F_{2\infty}(y) + \frac{u_*}{U_\infty}\left[F_{1\infty}(y^+) - B^{-1}Cy^{+\gamma}\right] \tag{39}$$

This composite solution has the following properties:

- As $\delta/\eta \to 1$, $U/U_\infty \to 1 + F_{2\infty}(\tilde{y})$. Thus there is a boundary layer profile even in the limit of infinite Reynolds number and it corresponds to the outer scaling law. This can be contrasted with the Millikan approach for which $U/U_\infty \to 1$ in this limit.
- As $\tilde{y} \to 0$, $U/U_\infty \to (u_*/U_\infty)F_{1\infty}(y^+)$ for all δ/η. This is because the small $\tilde{y}$ behavior of $[1 + F_{2\infty}(\tilde{y})]$ is cancelled out by the last term leaving only the inner solution.
- As $y^+ \to \infty$, $U/U_\infty \to 1 + F_{2\infty}$. This is because of the large y^+ behavior of $F_{1\infty}$ which is also cancelled by the last term.
- In the matched layer, only the power law profile remains.

It is an interesting exercise to substitute the composite solution into the full boundary layer equation given by equation 1. As expected, the equation reduces to equation 2 for infinite Reynolds number and to equation 3 as the wall is approached. This can be contrasted with the substitution of the Millikan law plus Coles wake function in which the outer equation vanishes identically.

7 The Displacement and Momentum Thicknesses

The displacement thickness, δ_*, is defined by

$$U_\infty \delta_* \equiv \int_0^\infty (U_\infty - U) dy \tag{40}$$

This can be expressed using equation 39 as

$$\frac{\delta_*}{\delta} = I_1 - I_2 \left(\frac{U_\infty \delta}{\nu} \right)^{-1} \tag{41}$$

or

$$\frac{\delta}{\delta_*} = I_1^{-1} + \frac{I_2}{I_1} \left(\frac{U_\infty \delta_*}{\nu} \right)^{-1} \tag{42}$$

where

$$I_1 \equiv - \int_0^\infty F_{2\infty}(\tilde{y}) d\tilde{y} \tag{43}$$

and

$$I_2 \equiv \int_0^\infty \left[F_{1\infty}(y^+) - B^{-1} C y^{+\gamma} \right] dy^+ \tag{44}$$

The integrals I_1 and I_2 are universal constants and can be evaluated from the data.

It is immediately obvious from equations 41 and 42 that the outer length scale is proportional to the displacement thickness. This is quite different from the Millikan theory where the displacement thickness vanishes relative to the unspecified outer length scale. The result here is thus consistent with the experimental observation that the velocity profiles can be collapsed in all but the region nearest the wall by the free stream velocity and the displacement thickness.

The momentum thickness, θ, is defined by

$$U^2 \theta \equiv \int_0^\infty U (U_\infty - U) dy \tag{45}$$

Again using equation 39, the result is

$$\frac{\theta}{\delta} = I_3 + I_2 \left(\frac{U_\infty \delta}{\nu} \right)^{\frac{1+2\gamma}{1+\gamma}} \tag{46}$$

where

$$I_3 \equiv -\int_0^\infty F_{2\infty}(\tilde{y})\,[1 + F_{2\infty}(\tilde{y})]\,d\tilde{y} \tag{47}$$

$$I_4 \equiv \int_0^\infty y^{+\gamma}\left[F_{1\infty}(y^+) - B^{-1}Cy^{+\gamma}\right]dy^+ \tag{48}$$

$$I_5 \equiv \int_0^\infty \left[F_{1\infty}(y^+) - B^{-1}Cy^{+\gamma}\right]^2 dy^+ \tag{49}$$

and

$$I_6 \equiv 2CB^{-\frac{1}{1+\gamma}} I_4 + B^{\frac{1}{1+\gamma}} I_5 \tag{50}$$

As for the displacement thickness, the momentum thickness is also asymptotically proportional to the outer length scale, but with a different constant of proportionality.

Thus, in the limit of infinite Reynolds number, the outer length scale is proportional to both of these dynamically significant length scales, in contrast to the Millikan theory. This will be seen to be more consistent with experiment than the Millikan result. The shape factor can be computed by taking the ratio of equation 41 and 46. The result is

$$\begin{aligned} H &\equiv \delta_*/\theta \\ &= -\frac{I_1}{I_3}\left\{1 - \frac{I_2}{I_3}\left(\frac{U_\infty\delta}{\nu}\right)^{-1} + B^{1/(1+\gamma)}\left[2B^{-1}C\frac{I_4}{I_3} - \frac{I_5}{I_3}\right]\left(\frac{U_\infty\delta}{\nu}\right)^{(1+2\gamma)/(1+\gamma)}\right\} \end{aligned} \tag{51}$$

The asymptotic shape factor is easily seen to be constant and given by

$$H \rightarrow -\frac{I_1}{I_3} \tag{52}$$

Note that the asymptotic shape factor is greater than unity since $F_{2\infty} \leq 1$ always since $I_1 < I_3$. This is in contrast to the Millikan result, but consistent with the experimental observations to date.

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