The POD and Its Application to the Axisymmetric Jet

William K. George
State University of New York at Buffalo
Buffalo, NY 14260 USA

POD (Proper Orthogonal Decomposition)techniques are currently in vogue to generate appropriate bases for dynamical systems models of turbulence (v Holmes *et al* 1996), but they have been used for more than 30 years to investigate coherent structures in turbulence (v George 1989). The problem was originally posed for turbulence by Lumley in the following manner: Suppose we have a random velocity field, $u_i(\cdot)$ where "$\cdot$" represents x_i, t or some subset of them. We seek to find a deterministic vector, say $\phi_i(\cdot)$ which has the maximum projection on u_i in a mean square sense; *ie* $\phi_i(\cdot)$ is chosen so that $< |u_i(\cdot)\phi_i(\cdot)|^2 >$ is maximized. The appropriate choice of $\phi_i(\cdot)$ can be shown by the calculus of variations to be given by

$$\int_{region} R_{ij}(\cdot,\cdot')\phi_j(\cdot')d(\cdot') = \lambda\phi_i(\cdot) \tag{1}$$

This is an integral equation for $\phi_i(\cdot)$ in which the kernel is given by the two-point correlation function, $R_{ij} =< u_i(\cdot)u_j(\cdot') >$. In general, equation 1 does not have a single solution but many, and their character depends on both the kernel *and* the region over which the integral is taken.

The most familiar application of the POD is to flows in which the region is of finite extent in one or more directions (or time), either naturally or because of artificially imposed boundaries. It is well-known that when the POD is applied to flows which are of finite total energy, then the classical Hilbert-Schmidt theory applies. In this case there are denumerably infinite POD modes (or eigenfunctions), and they are orthogonal. Thus the original velocity field can be reconstructed from them as

$$u_i(\cdot) = \sum_{n=1}^{\infty} a_n \phi_i^n(\cdot) \tag{2}$$

The *random* coefficients a_n are functions of the variables not used in the integral, and must be determined by projection back onto the velocity field; *ie*

$$a_n = \int_{region} u_i(\cdot)\phi^{*n}_i(\cdot)d(\cdot) \tag{3}$$

The eigenfunctions are ordered (meaning that the lowest order eigenvalue is bigger than the next, and so on) so the representation is optimal in the sense that the fewest number of terms is required to capture the energy.

Thus the POD has provided several insights and possibilities: First, because of the finite boundaries it has produced a *denumerably infinite* set of orthogonal functions which optimally (in a mean square sense) describe the flow. Second a finite subset of these functions can be used to produce a finite number of equations for analysis. This is accomplished by using them in a Galerkin projection of the governing equations (in our case the Navier-Stokes equations). Thus by truncating after a specified number of modes, the infinitely dimensional governing equations are reduced to a finite set (v Holmes *et al* 1996 for details).

Really interesting things happen to the POD if the flow is homogeneous or periodic. Note that, contrary to popular assumption (especially in the DNS and LES communities), these are *not* the same thing. The velocity field is said to be *periodic* in the variable x if $u(x) = u(x + L)$ where L is the period and the dependence on the other variables has been suppressed for now, as has the fact that the field is a vector. *Homogeniety*, on the other hand, means the statistics are independent of origin. For example, if a flow is homogeneous in a single variable, say x, then the two point correlation with separations in x reduces to $R(x,x') = \tilde{R}(r)$ where $r = x' - x$ is the separation. Note that by definition, homogeneous flows are not of finite total energy since they are of infinite extent, so the Hilbert-Schmidt theory cannot apply to them. Moreover, periodic fields are of finite total energy only if a single period is considered, since otherwise they repeat to infinity.

Now if periodicity and homogeniety are so different, why does the confusion arise? The POD provides the answer. For fields homogeneous in x, equation 1 can be shown to transform to

$$\int_{-\infty}^{\infty} \tilde{R}(r)\tilde{\phi}(x+r)dr = \tilde{\lambda}\tilde{\phi}(x) \tag{4}$$

Since the $\phi(x)$ on the right hand side is a function of x only, it can be included in the integral on the left. Since there is now no x-dependence left on the right hand side, it is immediately obvious that solution itself must eliminate the x-dependence on the left hand side. Therefore the eigenfunctions must be of the form $\tilde{\phi(x)} \sim \exp{(ikx)}$ where k is a wavenumber and *all* values of k are possible; *ie* $-\infty < k < \infty$. The coefficients, $\hat{u}(k)$, can be shown to be given by

$$\hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{-ikx}dx \tag{5}$$

and the velocity field can be reconstructed from them by

$$u(x) = \int_{-\infty}^{\infty} \hat{u}(x)e^{ikx}dk \tag{6}$$

Thus the POD for homogeneous fields reduces to the familiar *Fourier transform* which depends on the continuous variable k, so the number of eigenfunctions is *non-denumerable.*

The situation for periodic fields is almost the same, but not quite. Any periodic field, even a random one, can be represented by a *Fourier series*; *ie*

$$u(x) = \sum_{n=-\infty}^{\infty} a_n e^{i2\pi nx/L} \tag{7}$$

where the a_n are random and are determined in the usual manner. Using the orthogonality, the two-point correlation function can be written as

$$R(x,x') = \sum_{n=-\infty}^{\infty} < |a_n|^2 > e^{i2\pi n(x'-x)/L} \tag{8}$$

Thus the two-point correlation for periodic flows, like homogeneous flows, depends only on the difference variable $r = x' - x$. Hence the eigenvalue problem of the POD reduces to exactly the form of equation 4, except now the limits of integration are $(L/2, -L/2)$. It is easy to see that the POD modes must also be harmonic functions, like those for homogeneous flows. But there is a very important diffence which is obvious from the integral: for periodic flows the wavenumber must be given by $k = 2\pi n/L$ *and* n can only take integer values! The number of POD modes is now *denumerably infinite* instead of being *non-denumerable* (*ie* continuous in k). Moreover, *the POD modes and the Fourier modes are identical.* Thus the use of *Fourier series* to represent periodic fields is indeed optimal, at least in a mean square sense.

None of the approaches above applies to flows which are inhomogeneous, but of infinite extent (like most shear flows in the streamwise direction). In fact, it has not been at all clear until recently whether the POD integral even exists in all cases. Attempts to-date to apply the POD to the flow in these inhomogeneous directions have ended up applying the Hilbert-Schmidt theory to finite regions of the flow. And as a result, the eigenfunctions and eigenvalues found are dependent on the particular domain included in the decompostion. Clearly this is because it is the finite domain itself which is making the energy finite. Recently, however, Ewing and George 1995 were able to show that if similarity solutions of the *two-point* Reynolds stress equations were possible, then the POD could be applied *in similarity coordinates* and the eigenfunctions were harmonic functions in it.

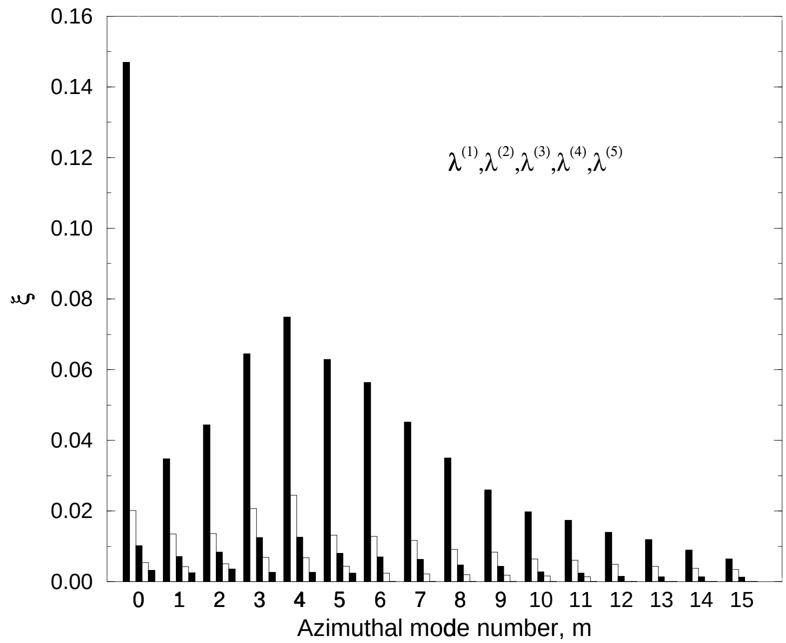

Figure 1: Azimuthal-mode energy distribution in the first 5 POD eigenvalues

In general real turbulent shear flows are homogeneous and/or periodic in some directions, and inhomogeneous in others. The near field of an axisymmetric jet has been of interest at SUNY/Buffalo for some time, and our recent experimental results include attempts to reconstruct the instantaneous velocity field at a single downstream location from the nozzle. The accompanying figure shows how the energy is distributed among the inhomogeneous POD modes for this flow and their azimuthal Fourier content. Note that many more azimuthal modes are required to represent the POD modes than vice versa, a consequence of the strong inhomogeniety across the flow, and the homogeniety around it. Current efforts are attempting to validate the analysis for the streamwise modes, and to determine the streamwise evolution of the POD modes. Recent and on-going work is summarized in the TRL web-site which includes several short movies indicating the behavior of partial re-compositions of the flow (http://www.eng.buffalo.edu/Research/trl/afosr or /trl/nsf)

REFERENCES

Ewing D and George WK (1995) Similarity Analysis of the Two-Point Velocity Correlation Tensor in the Turbulent Axisymmetry Jet. *Turbulence, Heat and Mass Transfer, Lisbon 1994* (Hanjalic and Pereira, eds.), Begell House Inc., NY, 49 – 56.

George WK (1989b) Insight into the Dynamics of Coherent Structures from a Proper Orthogonal Decomposition. in *Zorin Zaric Symp on Near-wall Turbulence, Dubrovnik, Yug* (S.Kline ed), Hemisphere, NY.