Effect of a Contraction on Turbulence. Part II: Theory*

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This paper considers the theoretical problems associated with the effect of a contraction on turbulence. The classical rapid distortion theory is first revisited and a new definition of rapid distortion will be provided that is more suitable for turbulence in a contraction. The implications of this definition will be explored in two different theoretical analyses: one using equilibrium similarity theory and the other using the single point Reynolds-stress equations. Both will be seen to yield solutions in which the turbulence intensities are powers of the local mean velocity along the centerline of the contraction. These theoretical findings will be verified using the experimental results presented in Part 1 of this study. The impact of these results on turbulence models will also be explored. An important finding is that the coefficient of the rapid pressure-strain correlation for differential stress models must be zero in the limit of rapid distortion.

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Nomenclature

$A,\ B$	Constants of proportionality
$B_{si,j}$	Time-dependent scaling functions for two point velocity correlations, m^2/s^2
$B_{sp,i}$	Time-dependent scaling functions for pressure-velocity correlations, kg/ms^3
$B_{i,j}$	Two-point velocity correlation tensor. m^2/s^2
$B_{p,i}$	Two-point pressure-velocity correlation, kg/ms^2
$B_{i,jk}, B_{ij,k}$	Triple velocity correlations, m^3/s^3
C_2	Parameter in rapid pressure strain-rate model
$I(t)$	Integral of strain-rate, m
k	Kinetic energy of turbulence, m^2/s^2
L_i	Integral scale in x_i-direction, m
M	Mesh size of grid, m
$m,\ n$	Constants
p	Fluctuating pressure, kg/m^2s^2
q^2	$\langle u^2 \rangle + \langle v^2 \rangle + \langle w^2 \rangle$ $(= 2k)$
r	Radial coordinate, m
$\mathcal{R}$	$\langle v^2 \rangle / \langle u^2 \rangle$
S	Mean strain rate, s^{-1}
t	Time, s
t_o	Virtual origin in time
U, u	Mean and fluctuating streamwise (or x) velocities, m/s
U_o	Mean velocity at grid
V, v	Mean and fluctuating radial velocities, m/s
V_c	Contraction volume as function of distance from exit
W, w	Mean and fluctuating velocity components in z direction
x	Streamwise coordinate, m
x'	Integration variable for streamwise coordinate
x_i	Coordinates ($i = 1, 2$, or 3)
β	Coefficient of proportionality
ν	Kinematic viscosity , m^2/s
ϵ	Rate of dissipation of turbulence energy per unit mass, m^2/s^3
ρ	Density, kg/m^3

I. Introduction

Contraction flows are commonly used in wind tunnel design, usually to accelerate the flow after it has been expanded through a diffuser in order to minimize pressure drop through flow manipulators like screens, honeycomb and heat exchangers (Uberoi,[1] Loehrke and Nagib[2]). The importance of studying contracting turbulence goes however far beyond windtunnel design. There are numerous engineering applications that involve contraction flows (e.g., turbomachinery and water turbines), and our ability to faithfully predict such flows becomes crucial in our quest for 'the optimal' design. Moreover, the solution to one class of flows has implications for others as well. Similar but 'reversed physics', for example, occurs in stagnation point flows and flows around bodies and airfoils, all of which are also dominated by strong strain-rate fields.

In Part 1 of this paper, Han et al.[3] demonstrate that the effect of the contraction on the unsteady flow must be considered in two parts: its effect on the coherent part and its effect on the turbulence part. It was demonstrated that all velocity components associated with the coherent part are amplified by the contraction and increase in proportion to the mean streamwise velocity, hence their magnitude relative to the mean flow remains constant throughout the contraction. By contrast, the contraction reduces the streamwise component of the turbulence part while amplifying the radial and azimuthal components. And finally, throughout most of the contraction the kinetic energy balance of the turbulence part along the centerline was almost entirely a balance between the mean convection of the turbulence and the production of turbulence energy by the working of the normal stress difference against the streamwise mean velocity gradient.

The flow along the centerline of an axisymmetric contraction is mathematically similar to homogeneously temporally varying strained turbulence. This feature makes this flow configuration particularly suitable for theoretical assessment. Homogeneously strained (and sheared) turbulence essentially forms the backbone for turbulence model formulations. There are numerous studies reported in the literature that are concerned with axisymmetric turbulence and its implications on turbulence modelling, only a few of which are Lee,[4] Sjögren and Johansson,[5] and Sambasivam et al.[6] The added complexity of structural disequilibrium due to the time varying background shear exhibited in this particular case makes it especially challenging. In fact, the majority of current models are calibrated against homogeneous turbulence in the structural equilibrium limit (e.g., Pope,[7] Gatski and Speziale,[8] and Johansson and Hallbäck[9]). The non-equilibrium characteristics of the flow through the contraction therefore makes the present study particularly valuable.

The objective of the present study is two-fold: firstly, to explore the theoretical consequences of the observed balance between mean convection and rate of production of turbulence kinetic energy; and secondly, to investigate the theoretical implications of these findings for turbulence model formulations. The paper is organized as follows. We will begin with a brief review of the classical rapid distortion theory and clarify some misconceptions about the manner in which is commonly used. In fact we will demonstrate that the classical approach is fundamentally flawed for this application. The implication of an equilibrium similarity approach (e.g. George[10]) is thereafter explored. Finally, we will explore the surprising consequences for the component Reynolds stress equations, and especially for models of the rapid pressure-strain correlation term.

II. Classical rapid distortion theory revisited

It has been customary since Batchelor[11] (see Durbin and Pettersson Reif[12] for a more comprehensive treatment) to use the results of linear rapid distortion theory (or RDT) applied to homogeneous turbulence to estimate the effect of a contraction on the upstream turbulence, e.g. Tucker and Reynolds.[13] This *linear* treatment leads directly to analytical relations showing how the turbulence intensities depend on time and the applied strain rate. The analysis begins by assuming a homogeneous turbulence which evolves in time when subjected to a uniform *and constant* strain rate field. These time-dependent theories have been applied to evolving flows (like contractions) by assuming the scale of the turbulence to be much smaller than the size of the contraction, and by assuming the turbulence to be convected along the flow so that time can be computed from the following equation

$$t - t_o = \int_{x_o}^{x} \frac{dx'}{U(x')}. \tag{1}$$

The final results provide ratios of local to upstream turbulence intensities as functions of the local mean velocity ratio (or area ratio inferred from streamtube equations). Other convenient results are the local turbulence anisotropy ratios, also as functions of the local mean velocity ratio.

Unfortunately the relatively simple analytical expressions of linear rapid distortion analysis have led many to apply to method to contraction flows. They cannot be, even approximately. The reason is that the underlying assumptions of rapid distortion theory (at least of the classical variety) preclude its application in most contractions of practical interest. In particular, the classical rapid distortion theory depends crucially on the assumption that the applied strain rate is constant, both in time and space, and thus not only that the magnitude of the shear rate parameter is sufficiently high. This, together with the assumption of small disturbances, allows the applied velocity field, $U_i(\mathbf{x})$, to be written as a linear combination of the position vector, x_i with constant coefficients, A_{ij}; i.e.,

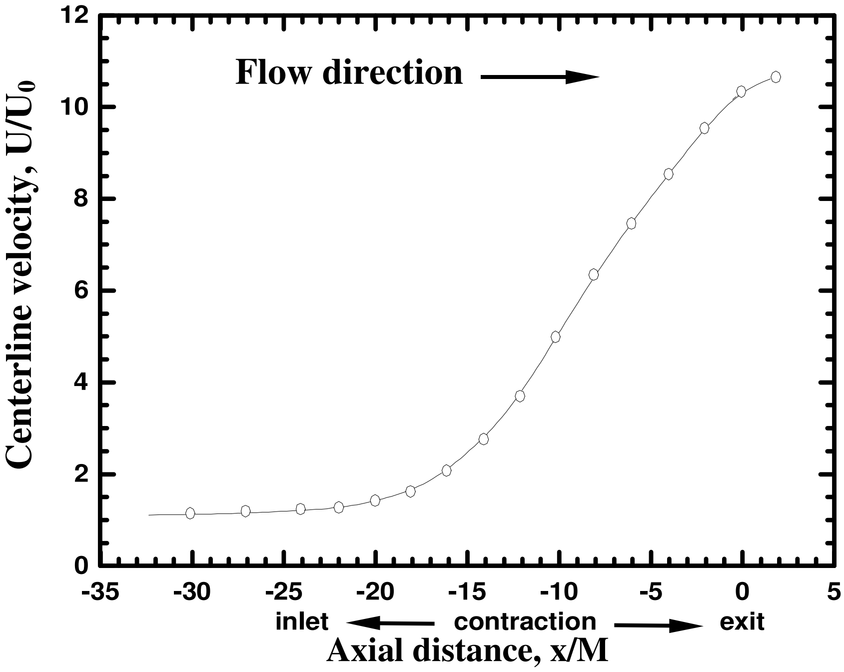

Figure 1. Centerline mean velocity through the 11.3:1 contraction of Han et al.[3] ($M = 0.0254$ m, $U_o = 2$ m/s)

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$$U_i(\mathbf{x}) = A_{ij} x_j. \tag{2}$$

One can certainly hypothesis flows that behave this way (like the classical two-dimensional irrotational sink or stagnation point flows). But this is not an accurate description of most three-dimensional contraction flows where the flow slowly accelerates at the entrance, then accelerates more rapidly as the area diminishes, and finally accelerates much less rapidly as the exit is approached. As a consequence the fundamental assumption of the classical approach to rapid distortion that the mean deformation tensor is constant is violated; therefore the exponential relations derived from it do not apply to these flows. An example of such a flow is illustrated by Figure 1 which shows the variation of the centerline velocity with distance in the contraction used by Han et al. [3] in Part 1 of this paper. Figures 2 and 3 show the streamwise velocity derivative plotted first as a function of distance along the centerline (measured from the exit plane), then as a function of time obtained from equation 1. Obviously the strain rate cannot be assumed constant over any range of interest, especially during the rising part of the curve where the assumptions of homogeneity are most likely to be valid.

The basic idea of rapid distortion is that the turbulence is stretched so rapidly that there is no time for the non-linear interactions to occur, one consequence of which is that there

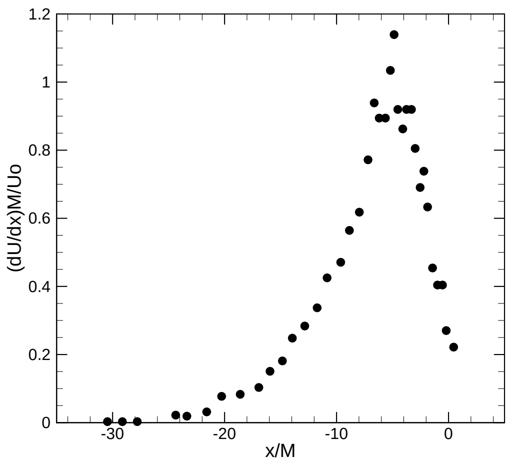

Figure 2. Centerline mean velocity derivative as function of distance through the contraction.

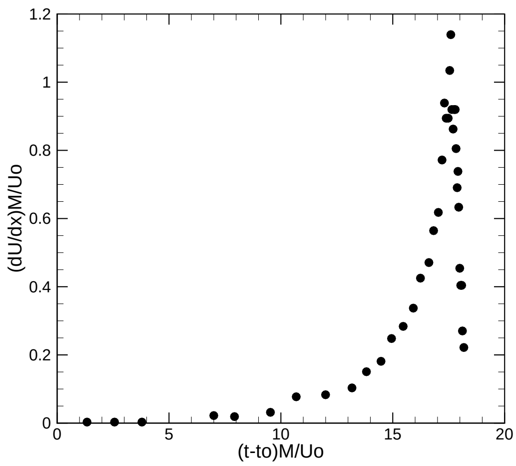

Figure 3. Centerline mean velocity derivative as function of time (using equation 1).

can be no significant dissipation of turbulent energy nor any turbulent transport. These are important constraints that should be carefully considered whenever rapid distortion theory is applied to measured or computed flow fields. Unfortunately, there has been considerable misunderstanding about this in past investigations of contracting turbulence where both transport and dissipation were playing a role, e.g. Sjögren and Johansson.[14]

It is clear from the above that we need to formulate a more general idea of 'rapid distortion' than in the classical analysis for contraction flows. We begin by considering the turbulent kinetic energy (k) equation, which for a high Reynolds number axisymmetric flow along the centerline can be shown to reduce to:

$$U\frac{\partial k}{\partial x} - [\langle u^2\rangle - \langle v^2\rangle]\frac{\partial U}{\partial x} = -\{\frac{\partial}{\partial x}[\frac{1}{\rho}\langle pu\rangle + \frac{1}{2}\langle q^2 u\rangle] + \frac{\partial}{\partial r}[\frac{2}{\rho}\langle pv\rangle + \langle q^2 v\rangle]\} - \epsilon \tag{3}$$

by using l'Hôpital's theorem and mass conservation. Note that $q^2 \equiv 2k$, and $\langle\rangle$ denotes ensemble average. The terms on the left-hand-side represent the convection by the mean flow and the production respectively, and the terms on the right-hand-side are the turbulence transport and dissipation. Obviously in any new theory, production and transport by the mean flow should dominate the kinetic energy equation, consistent with at least one of the ideas behind the classical theory. This can be illustrated by the kinetic energy balances (along the centerline) computed from the measurements presented in Part 1 of this paper, and which have been repeated here in Figure 4. In these three experiments the turbulence was generated by a grid upstream of the contraction. By changing the distance between the grid and the contraction over distances from approximately 20 to 120 grid mesh lengths, it was possible to vary the turbulence intensity and length scales of the turbulence entering the contraction. For case I (upper left), corresponding to the grid at only 24 mesh lengths upstream of the contraction entrance, the dissipation was negligible, but the transport terms were of becoming of importance by the middle of the duct ($x/M > -10$). For case II (upper right), corresponding to the grid at 44 mesh lengths upstream of the contraction entrance, the dissipation was negligible and the transport terms were also of some importance near the middle of the duct (but less so). And for case III, corresponding to the grid at 68 mesh lengths upstream of the contraction entrance, the dissipation and transport terms were even less important over entire duct.

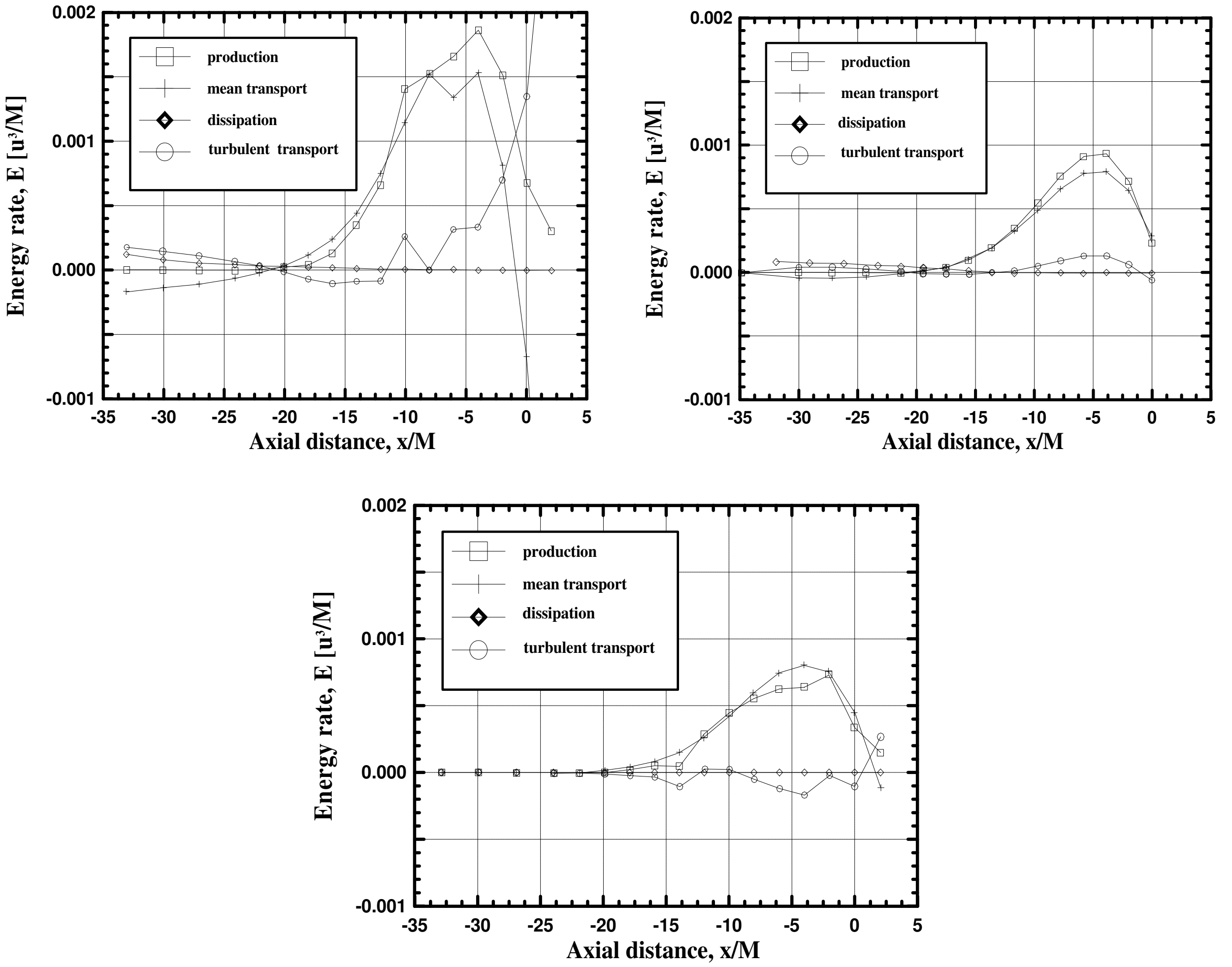

Figure 4. Upper left: grid at 24 mesh lengths upstream of contraction entrance. Upper right: grid at 44 mesh lengths. Lower: grid at 68 mesh lengths. The term labelled 'dissipation from energy balance' represents the net effect of all terms except production and mean transport. The measured dissipation is negligible in all cases.

Instead of using the two ideas of the classical rapid distortion theory (namely, linearized equations and the constancy of the deformation tensor), we will instead use only the Reynolds stress equations. Two approaches have been used: one working directly from the two-point Reynolds stress equations for a homogeneous turbulence subjected to a time-dependent strain-rate field; and the other using the single-point Reynolds stress equations. The former will prove particularly useful in understanding the physics and in setting the direction, the latter in understanding the implications for turbulence models.

III. Equilibrium similarity of the two-point Reynolds stress equations

The theoretical problem can be most easily posed by considering in Cartesian coordinates a homogeneous turbulence subjected to a time-dependent, but spatially uniform strain-rate field, $S(t) = \partial U/\partial x_1 = -2\partial V/\partial x_2 = -2\partial W/\partial x_3$. The 1,1 component of two-point Reynolds stress equations is given by:

$$\begin{aligned}\frac{\partial B_{1,1}}{\partial t} &+ Sr_1\frac{\partial B_{1,1}}{\partial r_1} - \frac{1}{2}Sr_2\frac{\partial B_{1,1}}{\partial r_2} - \frac{1}{2}Sr_3\frac{\partial B_{1,1}}{\partial r_3} \\ &= -2B_{1,1}S - \frac{1}{\rho}\frac{\partial}{\partial r_1}\left[B_{1,p} - B_{p,1}\right] + \frac{\partial}{\partial r_j}\left[B_{1,j1} - B_{1j,1}\right] + 2\nu\frac{\partial^2}{\partial r_j \partial r_j}B_{1,1}\end{aligned} \tag{4}$$

where $B_{1,1}(r_1, r_2, r_2, t) = \langle u_1(x_1, x_2, x_3, t)u_1(x_1 + r_1, x_2 + r_2, x_3 + r_3, t)\rangle$ with similar relations for the pressure velocity and triple velocity moments. Similar equations can be written for the 2,2 and 3,3 components.

We will seek an equilibrium similarity solution to these component Reynolds stress equations. We are interested only in solutions for which the viscous dissipation can be ignored — in fact we will 'define' this to be what we mean by the term 'rapid distortion'. It will not be necessary to assume anything about negligibility or non-negligibility of the non-linear terms. The basic approach can be outlined as follows: we seek solutions of the form:

$$B_{1,1}(r_1, r_2, r_3, t) = B_{s1,1}(t)f_{1,1}(\eta_1, \eta_2, \eta_3) \tag{5}$$

$$B_{p,1}(r_1, r_2, r_3, t) = B_{sp,1}(t)f_{p,1}(\eta_1, \eta_2, \eta_3) \tag{6}$$

etc ...

where $\eta_1 = r_1/L_1(t)$, $\eta_2 = r_2/L_2(t)$, $\eta_3 = r_1/L_3(t)$. All of the explicit time dependence is in the scaling parameters $B_{s1,1}(t)$, $B_{sp,1}(t)$, etc., and the length scales $L_1(t)$, $L_2(t)$, and $L_3(t)$.

These are substituted into the component Reynolds stress equations. Then it is demanded that all the terms (except the viscous term which has been neglected) have the same relative value at all times for given values of η_1, η_2, and η_3. This is the equilibrium similarity hypothesis.[15] It is relatively straightforward to show that such solutions are possible if the following conditions are satisfied:

$$\frac{\dot{L}_1}{L_1} \propto \frac{\dot{L}_2}{L_2} \propto \frac{\dot{L}_3}{L_3} \propto S(t) \tag{7}$$

$$\frac{\dot{B}_{s1}}{B_{s1}} \propto \frac{\dot{B}_{s2}}{B_{s2}} \propto \frac{\dot{B}_{s3}}{B_{s3}} \propto S(t) \tag{8}$$

with additional conditions for the triple velocity and pressure-velocity correlations. These will be ignored for now, but it should be noted that they are both non-zero, so non-linearity can remain part of the solution.

Now the conditions of equation (8) can be readily seen to imply that the logarithm of all of these quantities are proportional to the integral $I(t)$ defined by

$$I(t) \equiv \int_{t_o}^{t} S(t')dt', \tag{9}$$

which sometimes is referred to as the reference natural strain. In other words

$$\begin{aligned} \ln L_1(t)/L_1(0) &= \beta I(t) && (10)\\ \ln B_{s1,1}(t)/B_{s1,1}(0) &= mI(t) && (11)\\ \ln B_{s2,2}(t)/B_{s2,2}(0) &= nI(t) && (12)\\ &\text{etc ...} \end{aligned}$$

where β, m, n, etc. are constants which can at most depend on the upstream conditions, and must be determined either from experiment or other considerations.

In order to apply this theory to the experiments, it is necessary to relate the integral $I(t)$ to the variation of mean velocity in the contraction, i.e. $U = U(x)$ only, so $S = dU/dx$. Now if the turbulence intensity is small, then increments in time, dt, can be related to increments of space in exactly the manner of equation (1). (Note that a similar integral $I(t)$ occurs in the classical rapid distortion approach, but here without the requirement that S be constant.) Thus we have:

$$I(t) = \int_{t_o}^{t} \frac{dU}{dx}\frac{dx}{U} = \ln U/U_o \tag{13}$$

It follows immediately that all of the scaling quantities vary as powers of the mean velocity U; i.e.,

$$\begin{aligned} L_1(t)/L_1(0) &= (U/U_o)^{\alpha} && (14)\\ B_{s1,1}(t)/B_{s1,1}(0) &= (U/U_o)^{m} && (15)\\ B_{s2,2}(t)/B_{s2,2}(0) &= (U/U_o)^{n} && (16) \end{aligned}$$

etc ...

Moreover, the corresponding single point quantities, like $\langle u^2 \rangle$, $\langle v^2 \rangle$ and $\langle w^2 \rangle$ must likewise be simply proportional to powers of U.

IV. The component Reynolds-stress equations

The experiments presented in Part 1 certainly satisfy the assumptions of the rapid distortion equilibrium theory outlined above, at least until the scales grow to where the flow can no longer be assumed homogeneous and until the transport terms become important. But there are two additional problems, both related to the axisymmetric geometry. In the following paragraphs, a slightly different approach will be outlined using the single point Reynolds stress equations applied along the centerline of the flow. Like the two-point solutions above, the turbulence intensities will be seen behave as powers of U, at least as long as the assumptions are valid. But unlike the two-point approach, it will be possible to derive specific values for the exponents.

The transport of the kinematic Reynolds-stress components $\langle u_i u_j \rangle$ is governed by

$$\begin{aligned}\frac{\partial \langle u_i u_j \rangle}{\partial t} + U_k \frac{\partial \langle u_i u_j \rangle}{\partial x_k} \quad = \quad & \underbrace{-\left(\langle u_i u_k \rangle \frac{\partial U_j}{\partial x_k} + \langle u_j u_k \rangle \frac{\partial U_i}{\partial x_k} \right)}_{P_{ij}} \underbrace{- \nu \frac{\partial^2 \langle u_i u_j \rangle}{\partial x_k \partial x_k}}_{D^{\nu}_{ij}} \\ & \underbrace{-\frac{1}{\rho} \left(\langle p \frac{\partial u_i}{\partial x_j} \rangle + \langle p \frac{\partial u_j}{\partial x_i} \rangle \right)}_{\phi_{ij}} \underbrace{- 2\nu \langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \rangle}_{\varepsilon_{ij}} \\ & + \underbrace{\frac{1}{\rho} \left(\frac{\partial}{\partial x_i} \langle u_j p \rangle + \frac{\partial}{\partial x_j} \langle u_i p \rangle \right)}_{D^{p}_{ij}} \underbrace{- \frac{\partial}{\partial x_k} \langle u_i u_j u_k \rangle}_{D^{t}_{ij}} \end{aligned} \tag{17}$$

where P_{ij} and D^{ν}_{ij} are the rate of production due to mean shear and viscous diffusion, respectively. The remaining terms on the right-hand-side comprises the pressure-strain (ϕ_{ij}), rate of viscous dissipation ε_{ij}, and pressure (D^{p}_{ij}) and turbulent (D^{t}_{ij}) diffusion, respectively. Cartesian tensor notation is used here and $[x_1, x_2, x_3]$ denote the axial, radial, and azimuthal directions.

By using l'Hôpital's rule and the continuity equation, the diagonal components of $\langle u_i u_j \rangle$ along the centerline (neglecting the viscous transport and dissipation terms) can be shown to be given by:

• Axial (centerline):

$$0 = -U\frac{\partial}{\partial x}[\frac{1}{2}\langle u^2\rangle] - \langle u^2\rangle\frac{\partial U}{\partial x} - \langle\frac{p}{\rho}\frac{\partial u}{\partial x}\rangle - \{\frac{\partial}{\partial x}[\frac{1}{\rho}\langle pu\rangle + \frac{1}{2}\langle u^3\rangle] + \frac{\partial}{\partial r}[\langle u^2 v\rangle]\} \quad (18)$$

• Radial (centerline):

$$0 = -U\frac{\partial}{\partial x}[\frac{1}{2}\langle v^2\rangle] + \frac{1}{2}\langle v^2\rangle\frac{\partial U}{\partial x} + 2\langle\frac{p}{\rho}\frac{\partial v}{\partial r}\rangle - \{\frac{\partial}{\partial x}[\frac{1}{2}\langle uv^2\rangle] + \frac{\partial}{\partial r}[\frac{2}{\rho}\langle pv\rangle + \langle v^3\rangle - \langle v^2 w\rangle]\} \quad (19)$$

• Azimuthal (centerline):

$$0 = -U\frac{\partial}{\partial x}[\frac{1}{2}\langle w^2\rangle] + \frac{1}{2}\langle w^2\rangle\frac{\partial U}{\partial x} + \langle\frac{\partial}{\partial r}[\frac{p}{\rho}\frac{\partial w}{\partial \theta}]\rangle - \{\frac{\partial}{\partial x}[\frac{1}{2}\langle uw^2\rangle] + \frac{\partial}{\partial r}[\frac{2}{\rho}\langle pw\rangle + 2\langle w^2 v\rangle]\} \quad (20)$$

The primary difference from the kinetic energy equation is the presence of the pressure-strain rate terms which can move energy from one component to another ($\phi_{ii} = 0$). As before, the viscous transport terms have been neglected, and the turbulence transport terms (bracketed $\{\}$) are usually negligible also.

Since the turbulence transport terms are usually negligible, it is readily seen that where there is no *streamwise* mean velocity gradient (as in most of the duct flow), the energy is simply redistributed among the various diagonal components of the Reynolds-stress tensor, dissipated, or convected away by the mean velocity. When the mean flow is accelerating in the contraction, however, it follows immediately that there is *positive* production of the radial and azimuthal components since $\partial U/\partial x > 0$. By contrast, the "production" term in the streamwise component equation is *negative*, however, implying that this term actually acts to *decrease* $\langle u^2\rangle$. This explains the observed reduction of the streamwise normal stress component in Part 1 (at least after the coherent disturbances were removed).

A. A power law solution

The equilibrium similarity solutions derived above represent very special forms of the equations after the flow has evolved to an equilibrium state. These should not be expected to apply in general, since the flow may be evolving too rapidly for equilibrium to have been achieved. In fact, the flow is by its very definition in a state of structural disequlibrium, since the mean strain rate varies in time. True structural equilibrium is only obtained if the turbulence anisotropies ($\langle u^2\rangle/k$,$\langle v^2\rangle/k$, and $\langle w^2\rangle/k$), and the turbulent-to-mean-flow time scale ratio, Sk/ε, are constant. Although both of these constraints are violated here it is possible to show that certain features of these solutions are more general by considering only the kinetic energy equation and assuming there is a balance between advection and mean shear production as discussed earlier; i.e.,

$$U\frac{dk}{dx} = \left(\langle v^2\rangle - \langle u^2\rangle\right)\frac{dU}{dx}. \tag{21}$$

Along the centerline of the contraction, the turbulence is axisymmetric so that $k = (\langle u^2\rangle + 2\langle v^2\rangle)/2$ such that $Udk/dt = U/2d\langle u^2\rangle/dt + Ud\langle v^2\rangle/dt$. Solutions are now sought of the form $\langle u^2\rangle = AU^m$ and $\langle v^2\rangle = BU^n$ where A and B are constants possibly dependent on upstream conditions. Substitution yields:

$$\left(\frac{mA}{2}U^m + nBU^n\right)\frac{dU}{dx} = \left(BU^n - AU^m\right)\frac{dU}{dx} \tag{22}$$

or equivalently,

$$\frac{A}{B}U^{m-n}\left(1 + \frac{1}{2}m\right) = (n-1)\,. \tag{23}$$

Since the right hand side is independent of velocity, this equation has two possible solutions:

$$(i) \quad m = n = \tfrac{A-B}{A/2+B} \tag{24}$$

$$(ii) \quad m = -2, \quad n = 1, \tag{25}$$

but only the latter is viable (i.e., $\langle u^2\rangle = AU^{-2}$ and $\langle v^2\rangle = BU$) since the turbulence is anisotropic in general through the contraction. This solution can be shown to be consistent with the exact result $b_{22}/b_{11} \equiv -1/2$ in axisymmetric turbulence, where $b_{ij} = \langle u_i u_j\rangle/(2k) - \delta_{ij}/3$ denotes the Reynolds-stress anisotropy tensor, c.f., Lee.[4]

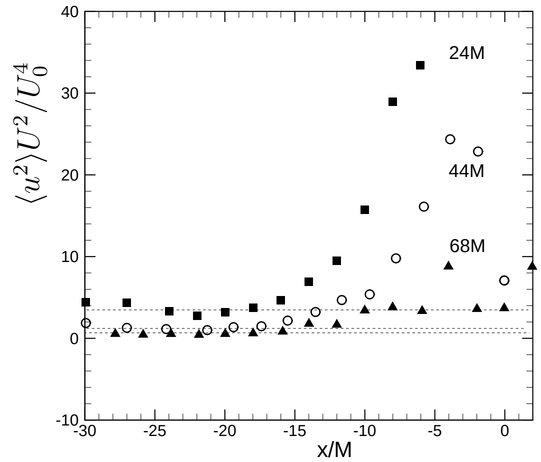

Figure 5. $\langle u^2\rangle$ times U^2 versus x/M. 24M, 44M, and 68M represent the different grid positions.

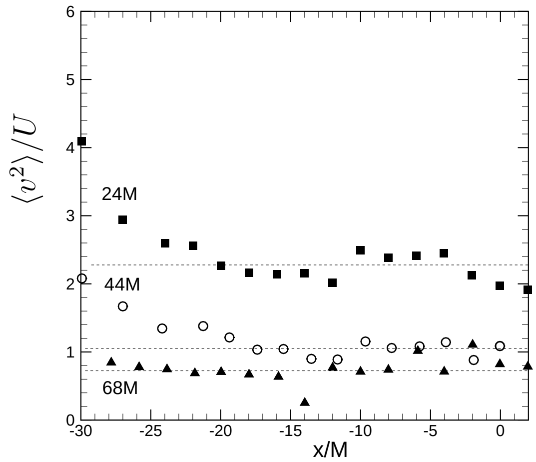

Figure 6. $\langle v^2\rangle$ divided by U versus x/M. 24M, 44M, and 68M represent the different grid positions.

It should be noted that the power-law behavior is only a consequence of the assumed balance between the advection and rate of mean-shear production of turbulent kinetic energy. Figures 5 and 6 show plots of $\langle u^2 \rangle U^2$ and $\langle v^2 \rangle / U$ versus distance through the contraction for the three experiments reported in Part 1. The plot for $\langle u^2 \rangle$ is approximately constant until $x/M < -10$, which is approximately where the transport terms begin to be important in the kinetic energy balances of figure 4. Moreover, the weaker the transport terms, the closer the data are to the power law behavior. For $\langle v^2 \rangle$, however, the power law solution is valid for the entire duct ($x/M > -24$). This is consistent with the fact that $\langle v^2 \rangle$ increases through the contraction so that its production term increases relative to the neglected transport terms. The opposite occurs for $\langle u^2 \rangle$, so the approximation deteriorates rapidly beyond $x/M > -10$.

V. Implications for turbulence modelling

The power-law solution for the individual diagonal components of the Reynolds-stress tensor can now used *a priori* to evaluate different turbulence model formulations. In the present case, the components of the mean rate-of-strain and mean vorticity tensors are given by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) = \left\{ \begin{array}{ccc} S & 0 & 0 \\ 0 & -S/2 & 0 \\ 0 & 0 & -S/2 \end{array} \right\}, \tag{26}$$

and

$$W_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) = \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right\}, \tag{27}$$

respectively. Before full differential stress models are considered, a brief analysis of two different types of eddy-viscosity models will be carried out. These do not depend on the power-law solution *per se*, but do illustrate some characteristic features of algebraic stress models in general.

A. Linear algebraic Reynolds-stress models

The linear eddy-viscosity model is essentially based on Boussinesq's proposal more than a century ago, and is still the most commonly used closure model in applied computational fluid dynamics. (The standard $k - \varepsilon$ model, among others, utilizes this simple constitutive relationship.) The hypothesis is essentially based on the notion that there is a linear algebraic relation between the turbulent stresses and the mean flow field:

$$\langle u_i u_j \rangle = \frac{2}{3} k \delta_{ij} - 2C_\mu \frac{k^2}{\varepsilon} S_{ij}. \tag{28}$$

The terminology 'linear' alludes to the linear dependence of $\langle u_i u_j \rangle$ on the mean rate of strain tensor S_{ij}. The diagonal components of the Reynolds-stress tensor are thus explicitly given by

$$\langle u^2 \rangle = \frac{2}{3} k - 2C_\mu k \frac{Sk}{\varepsilon} \tag{29}$$

$$\langle v^2 \rangle = \langle w^2 \rangle = \frac{2}{3} k + C_\mu k \frac{Sk}{\varepsilon}. \tag{30}$$

Symmetry along the centerline of the contraction implies that $\langle w^2 \rangle \equiv \langle v^2 \rangle$. $C_\mu = 0.09$ is the most frequently used value. Using this and imposing the realizability requirements $\langle u^2 \rangle \geq 0$ and $\langle v^2 \rangle = \langle w^2 \rangle \leq k$ in (29) and (30) respectively yields:

$$\frac{Sk}{\varepsilon} \leq \frac{1}{3C_\mu} \approx 3.7. \tag{31}$$

This result shows that only a very mild rate of straining can be represented by this simple model. Clearly it can not hope to represent the physics at the values of Sk/ϵ commonly encounter in contraction flows of the type considered in Part 1 (typically 30 - 50).

B. Explicit algebraic Reynolds-stress models

The next level of models within the eddy-viscosity framework are nonlinear constitutive models. The terminology 'nonlinear' refers to a nonlinear dependence of the Reynolds stresses on the mean strain rate. There exist basically two different classes of nonlinear closures and these are defined by the origin of the model: (i) the algebraic stress approximation of the Reynolds-stress transport models (so-called EASM models), and (ii) *ad hoc* nonlinear constitutive modeling using tensor calculus (NLEVM's). The former class of models referred to as explicit algebraic Reynolds-stress models (EASM) has experienced a growing popularity among CFD practitioners. The terminology 'explicit' is sometimes misleading and perhaps redundant since all eddy-viscosity models are explicit in the stresses. The reason EASM-models have gained popularity is the fact that these are *exact* solutions of the more sophisticated differential stress-models in the limit of equilibrium homogeneous turbulence (c.f., Pope[7]). As such these models are believed to encompass 'more physics' than *ad hoc* NLEVM's. It should be pointed out, however, that there exists a formal link between differential stress models and EASM's *only* in two-dimensional mean flow fields. The crucial assumption of structural equilibrium that needs to be invoked in order to solve the differential stress equations ceases to be valid in three-dimensional flows. Three-dimensional flow can not evolve to a structural equilibrium state; therefore the formal applicability of these models is limited to two-dimensional flows, c.f., Durbin and Pettersson Reif.[16] Three-dimensional extensions of the the original EASM (e.g., Taulbee;[17] Gatski and Speziale;[8] Johansson & Hallbäck[9]) are thus only *ad hoc*.

The following analysis is concerned only with models that are based on the equilibrium solutions of the differential stress models. The most general constitutive realationship (which thus is two-dimensional) can be written as:

$$\frac{\langle u_i u_j \rangle}{k} = \frac{2}{3}\delta_{ij} - 2C_\mu^* \left[\frac{kS_{ij}}{\varepsilon} + \alpha_2 \frac{k^2}{\varepsilon^2} \left(S_{ik}W_{kj} - W_{ik}S_{kj} \right) - 2\alpha_1 \frac{k^2}{\varepsilon^2} \left(S_{ik}S_{kj} - \frac{1}{3} S_{km}S_{km}\delta_{ij} \right) \right] \tag{32}$$

where the coefficient can be written as:

$$C_\mu^* = \frac{\alpha_0}{1 - \frac{2}{3}\eta_1 - 2\eta_2} = \frac{\alpha_0}{1 - \alpha_1^2 (Sk/\varepsilon)^2}. \tag{33}$$

The second equality is obtained by substituting $\eta_1 \equiv (k/\varepsilon)^2 (S_{km}S_{km}) = \frac{3}{2}\alpha_1^2 (Sk/\varepsilon)^2$ and $\eta_2 \equiv (k/\varepsilon)^2 (W_{km}W_{km}) = 0$ for the present case. The model constants are $\alpha_0 = 0.0567$ and $\alpha_1 = 0.0437$ for a given differential stress model (v. Gatski and Speziale[8] for the details).

The normal-stress components are given by

$$\frac{\langle u^2 \rangle}{k} = \frac{2}{3} - 2\alpha_0 \left(\frac{Sk}{\varepsilon} \right) \frac{1 - \alpha_1 (Sk/\varepsilon)}{1 - \alpha_1^2 (Sk/\varepsilon)^2} = \frac{2}{3} - \frac{2\alpha_0 (Sk/\varepsilon)}{1 + \alpha_1 (Sk/\varepsilon)} \tag{34}$$

$$\frac{\langle v^2 \rangle}{k} = \frac{\langle w^2 \rangle}{k} = \frac{2}{3} + \alpha_0 \left(\frac{Sk}{\varepsilon} \right) \frac{1 - \alpha_1 (Sk/\varepsilon)}{1 - \alpha_1^2 (Sk/\varepsilon)^2} = \frac{2}{3} + \frac{\alpha_0 (Sk/\varepsilon)}{1 + \alpha_1 (Sk/\varepsilon)}. \tag{35}$$

Imposing the realizability constraints $\langle u^2 \rangle \geq 0$ and $\langle v^2 \rangle \leq k$ in equations (34) and (35) respectively yields the combined constraint:

$$\frac{Sk}{\varepsilon} \leq \frac{1}{3\alpha_0 - \alpha_1} \approx 7.9. \tag{36}$$

Thus although the EASM model is able to handle somewhat higher strain rates than the linear eddy-viscosity model, it too fails completely to represent the experimentally observed shear rate ($Sk/\varepsilon \approx 50$).

Thus it is clear that algebraic stress models tend quickly to become unrealizable as the imposed straining increases. This problem has been recognized by Durbin[18] who devised a simple upper bound on the eddy-viscoity for linear models to avoid the problem in practical computations (in fact this bound correspond precisely to the inequality of equation (31). The constraint given by equation (36) provides the corresponding bound for the explicit algebraic stress model.

A comparison between the constitutive relations of equations (29) and (34) shows that the linear eddy-viscosity model can be viewed as a special case of the more elaborate explicit algebraic stress model in the limit as $Sk/\varepsilon \ll 1$; the only difference being that the constants are slightly different ($C_\mu = 0.09$ and $\alpha_0 = 0.0567$, respectively). Since the EASM was derived from a differential stress model under the assumption of structural equilibrium ($Sk/\varepsilon \sim 6$ in homogeneous shear) the present study elucidates the difficulties associated with applying a model to a problem that exhibits different characteristics than those used to calibrate the model in the first place.

C. Differential Reynolds-stress models

Turbulence models based on the differential Reynolds-stress approach constitute the most advanced level of RANS modelling currently used in practice, and it is a physically significantly more appealing methodology than algebraic constitutive models. In particular, differential stress models account exactly for the total rate of change of the turbulent stresses, the rate of production of the individual Reynolds-stress components by the mean strain, as well as for the viscous diffusion. No modelling is necessary for these terms; i.e., the terms on the left-hand-side of equation (17) and the two first on the right-hand side. While the latter of these terms is relatively unimportant, the two former terms are crucial.

Models need, however, to be provided for the remaining terms of equation (17), i.e., the pressure-strain correlation, the viscosus dissiaption rate tensor, and the turbulent transport terms. The latter can be neglected here. The pressure strain-rate terms are usually modelled in two parts: the rapid part denoted by ϕ_{ij}^R, and the slow part (return-to-isotropy) denoted by ϕ_{ij}^S. The most commonly used pressure strain-rate models are the IP (or isotropization-of-production) model and Rotta's linear return-to-isotropy given by:

$$\phi_{ij}^R = -C_2\left(P_{ij} - \frac{1}{3}P_{kk}\delta_{ij}\right), \quad \phi_{ij}^S = -C_1\varepsilon\left(\frac{\langle u_i u_j\rangle}{k} - \frac{2}{3}\delta_{ij}\right). \tag{37}$$

The rate of dissipation is assumed to be isotropic; i.e.,

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} \tag{38}$$

and this term is included for completeness. Using these, the transport equations governing the two independent Reynolds-stress components can be written as:

$$U\frac{d\langle u^2\rangle}{dx} = \left[-2(1-C_2)\langle u^2\rangle + \frac{2}{3}C_2(\langle v^2\rangle - \langle u^2\rangle)\right]\frac{dU}{dx} + \frac{2}{3}\varepsilon(C_1 - 1) - C_1\varepsilon\frac{\langle u^2\rangle}{k} \tag{39}$$

$$U\frac{d\langle v^2\rangle}{dx} = \left[(1-C_2)\langle v^2\rangle + \frac{2}{3}C_2(\langle v^2\rangle - \langle u^2\rangle)\right]\frac{dU}{dx} + \frac{2}{3}\varepsilon(C_1-1) - C_1\varepsilon\frac{\langle v^2\rangle}{k} \tag{40}$$

Dividing the first by $\langle u^2\rangle U$, the second by $\langle v^2\rangle U$, and then subtracting the former from the latter yields:

$$\frac{d\ln\mathcal{R}}{dx} = \frac{d\ln U}{dx}\left(3(1-C_2) - \frac{2}{3}C_2(\mathcal{R}^{-1}-\mathcal{R}) - \frac{2}{3}(C_1-1)\frac{\varepsilon}{Sk}\frac{k}{\langle v^2\rangle}(1-\mathcal{R})\right) \tag{41}$$

where $\mathcal{R} \equiv \langle v^2\rangle/\langle u^2\rangle$. If we further incorporate the power-law solution $\mathcal{R} = DU^\alpha$ where $\alpha = n-m$, and $D = A/B$, the equation reduces to:

$$\frac{d\ln U}{dx}\left(\alpha - 3(1-C_2) + \frac{2}{3}C_2(\mathcal{R}^{-1}-\mathcal{R}) + \frac{2}{3}(C_1-1)\frac{\varepsilon}{Sk}\frac{k}{\langle v^2\rangle}(1-\mathcal{R})\right) = 0 \tag{42}$$

Since from the above, $m=-2$ and $n=1$, $\alpha = n-m = 3$. Therefore:

$$C_2 = \frac{2}{3}(C_1-1)\left[\frac{\varepsilon}{Sk}\right]\left[\frac{k}{\langle v^2\rangle}\right]\frac{(1-\mathcal{R})}{3+\frac{2}{3}(\mathcal{R}^{-1}-\mathcal{R})} \tag{43}$$

There are a number of interesting observations that can be made about the rapid coefficient $C_2 = C_2(\mathcal{R}, C_1, Sk/\varepsilon)$ which commonly takes the value $C_2 = 0.6$. In particular:

1. It cannot take a constant non-zero value. Rather it should depend explicitly on the turbulence anisotropy and vanish identically in isotropic turbulence; i.e., when $\mathcal{R} = 1$;
2. It is related to the slow-pressure strain model through the coefficient C_1;
3. It should depend inversely on the mean shear rate, Sk/ε, and its limiting value must be zero in the limit as $Sk/\varepsilon \to \infty$.

The latter dependence has been by used by Lee,[4] for example, to develop a simple model for the rapid pressure-strain correlation. It has in fact been rather common to make the slow pressure-strain coefficient depend on the time scale ratio, Sk/ε. For example,, Speziale et al.[19] adopt $C_1 \sim P/\varepsilon \sim (Sk/\varepsilon)^2$. Sjögren and Johansson[5] also speculate about incorporating the time-scale ratio in C_1 to account for the suppression of the return-to-isotropy at high shear rates. Most interesting in this context, however, is that in the rapid shear rate limit ($\varepsilon/Sk \to 0$) the limiting value of C_2 must be exactly zero. This surprising behavior originates from our demand that the model must be consistent with the theoretically derived (and experimentally observed) power-law behavior of the Reynolds-stresses through the contraction. It should also be emphasized that the result of equation (43) is preliminary; its application for practical computations is outside the scope of the present study. Even so it certainly provides an indicator of the direction improvements to the models might take.

Summary and conclusions

It has been demonstrated that the classical rapid distortion theory does not apply to contraction flows since the basic assumption of the theory is not satisfied. In particular, the mean deformation tensor cannot be assumed constant for contraction flows. An alternative definition of rapid distortion has been provided in which it is assumed that the dissipation and turbulence transport terms are negligible. No assumptions about the deformation rate nor even linearization were required.

The new definition was applied in two ways, both of which lead to solutions in which the turbulence intensities were proportional to powers of the mean velocity. The assumption of equilibrium similarity also provided relations governing the evolution of the two-point pressure velocity and triple velocity moments, as well as the integral scales. These were not tested against experimental data, but could most easily be tested using DNS. This was however not within the scope of this paper. Application of the new definition to the kinetic energy equation provided power law solutions in which the streamwise component of the Reynolds stress varied inversely with the square of the mean velocity, while the radial component varied linearly with the mean velocity. These showed good agreement with the experimental data presented in Part 1.

The implications of contracting turbulence on various turbulence models ranging from the linear eddy-viscosity model to a commonly used differential stress model were also explored. It was shown firstly by *a priori* considerations that the application of algebraic stress models was seriously limited to only small imposed shear rates and therefore not suitable for the present case.

Finally, a commonly used differential stress model were used to investigate the implications of the theoretically derived, and experimentally observed power-law behaviour of the Reynolds-stresses. It was shown that the only possible constant value of the rapid pressure-strain model was zero. The analysis also showed that in order to retain a nonzero value, the model coefficient must depend explicitely on the turbulence anisotropy as well as on the shear rate. Further, it was shown that the rapid and slow pressure-strain model coefficients can not be determined independently. Most interesting perhaps was the conclusion that the only possible value of the rapid pressure strain-rate coefficient in the rapid distortion limit is zero.

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